	Journal Code				Article ID				Dispatch: 05.10.13	CE: Mary Charlane Payuan
SPi	C	T	A		1	9	5	6	No. of Pages: 14	ME:

INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS
Int. J. Circ. Theor. Appl. (2013)
Published online in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/cta.1956

Unified theory and state-variable implementation of critical-monotonic all-pole filters

Dragan Topisirović[1], Vančo Litovski[2] and Miona Andrejević Stošović[2,*,†]

[1]*Regional Centre for Talents, 18000 Niš, Serbia*
[2]*University of Niš, Faculty of Electronic Engineering, 18000 Niš, Serbia*

SUMMARY

The subject of synthesis of critical-monotonic low-pass amplitude characteristics will be revisited. Several new contributions will be given in order to: facilitate the choice of the proper transfer function, to allow cataloguing the transfer functions, to simplify the circuit synthesis procedure, and to perform synthesis in the form of a state-variable continuous time active filter. Four main criteria for transfer function synthesis will be implemented: maximally flat at the origin, maximum slope at the band-edge, maximal asymptotic attenuation, and minimal amplitude distortion in the pass-band. For every criterion, a class of filters will be generated and the coefficients of the transfer functions will be calculated and published for the first time (with one exception). Properties of the classes so generated will be quantitatively compared for the first time. The state-variable structure will be advised as the one with the simplest synthesis procedure. The procedure will be explained and the design process will be exemplified. Statistical tolerance analysis will be performed for the example solutions in order to complete the information for comparison. Copyright © 2013 John Wiley & Sons, Ltd.

Received 25 September 2012; Revised 22 April 2013; Accepted 21 September 2013

KEY WORDS: electronic filters; approximation; polynomials; least-squares; transfer function; state-variable filters

1. INTRODUCTION

There is more than 80 years since the first all-pole monotonic amplitude filtering characteristic was published by Butterworth [1]. Since it has all its derivatives in the origin equal to zero, it theoretically represented the only alternative to the non-monotonic characteristic using the Chebyshev polynomials as characteristic function. The simplicity of expressing the criterion implemented for its derivation and the simplicity of its characteristic function was probably the only reason why this solution was, is, and will still be so popular in the future.

Searching modern literature, especially textbooks, one gets the feeling as if no alternative monotonic solutions were developed in the meantime. In fact, the situation is different and a large set of monotonic all-pole functions were published outperforming the Butterworth solution in every respect. Implementation of some of these solutions may lead to serious benefits to the designer and producer. It is our goal here to revisit the subject. We intend first to list the main criteria implemented for synthesis of all-pole functions with monotonic amplitude. Then, we intend, for the first time, to express a unified theory that covers the creation of the main solutions published mainly in the early 70s and before. It is our intention to give qualitative and quantitative comparisons of the properties of the main classes of monotonic all-pole filters. A small catalogue of the transfer function denominator coefficients will be given which apart from the Butterworth filters was not available in the literature. We will also propose

*Correspondence to: Miona Andrejević Stošović, University of Niš, Faculty of Electronic Engineering, A. Medvedeva 14, 18000 Niš, Serbia.
†E-mail: miona.andrejevic@elfak.ni.ac.rs

a systematic yet effective method for implementation of the all-pole functions in a form of state-variable filters. Finally, we intend to demonstrate the method of monotonic filter design and some properties of the circuit realization. In that way, we hope, we will deliver a platform offering more design freedom and better solutions.

2. THE CRITICAL-MONOTONIC ALL-POLE AMPLITUDE CHARACTERISTIC

The squared modulus of the amplitude characteristic of a low-pass filter may be written as

$$|T(j\omega)|^2 = \frac{1}{1+\varepsilon^2 K(\omega^2)} \tag{1}$$

where ε^2 is a constant used to control the amplitude characteristic at the edge of the pass-band and for now will be considered equal to unity, while $K(\omega^2)$ is the characteristic function of the filter. In the case of an all-pole filter, it is an even polynomial of the angular frequency ω. Halpern [2] proposed to write it in the following form:

$$K(\omega^2) = L_n(\omega^2) = \int_0^{\omega} x \cdot E_{n-1}(x^2) \cdot dx. \tag{2}$$

Since here we are looking for a monotonic amplitude characteristic, $E_{n-1}(x^2)$ is to be chosen so that it enables critical monotonicity. The first derivative of a critical-monotonic function never changes its sign. The first condition for that is $E_{n-1}(x^2)$ to be a full square, i.e. to be expressed as a square of another polynomial: $E_{n-1}(x^2) = V_{n-1}^2(x)$, where $V_{n-1}(x)$ is to be an odd or an even polynomial. The second condition was that all the zeros of $V_{n-1}(x)$ have to be real and to be located in the interval {0, 1}. To that end, $V_{n-1}(x)$ was expressed as a sum of orthogonal polynomials with the interval of orthogonality defined by the normalized pass-band of the filter, i.e. ω∈{0, 1}.

To avoid repeating the proper developments here, we are giving the expressions of $V_{n-1}(x)$ as [2]:

$$V_{n-1} = \sum C_i \cdot U_i(x) \tag{3}$$

where C_i are properly chosen constants, and $U_i(x)$ are Jacobi polynomials satisfying the following relation

$$\int_0^1 x \cdot U_j \cdot U_k \cdot dx = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \text{for } j = k \end{cases} \tag{4}$$

while j and k are both even or both odd natural numbers. In that way, we get

$$L_n(\omega^2) = \int_0^{\omega} x \left\{ \sum_{i=0}^{(n-2)/2} C_{2i+1} \cdot U_{2i+1}(x) \right\}^2 \cdot dx \qquad \text{for } n\text{-even}, \tag{5}$$

and

$$L_n(\omega^2) = \int_0^{\omega} x \left\{ \sum_{i=0}^{(n-1)/2} C_{2i} \cdot U_{2i}(x) \right\}^2 \cdot dx \qquad \text{for } n\text{-odd}, \tag{6}$$

where n is the order of the filter.

The Jacobi polynomials are defined by

$$U_{2i+1}(x) = 2\sqrt{i+1} \sum_{m=0}^{i} (-1)^m \frac{(2i+1-m)!}{m! \cdot (i+1-m)! \cdot (i-m)!} x^{2(i-m)+1} \text{ for } n\text{-even}, \tag{7}$$

and

DOI: 10.1002/cta

$$U_{2i}(x) = \sqrt{4i+2}\sum_{m=0}^{i}(-1)^{i-m}\frac{(i+m)!}{(m!)^2\cdot(i-m)!}x^{2m} \qquad \text{for } n\text{-odd}. \tag{8}$$

Table I contains the first 10 Jacobi polynomials. Finally, the constants C_i have to be chosen in that way the normalization criterion: $L_n(1) = 1$ is satisfied, i.e. T1

$$\sum_k C_k^2 = 1. \tag{9}$$

One should mention that in (3) in the place of U_i, one may use any other classes of orthogonal polynomials having simple zeroes in the interval {0, 1}.

The functions expressed by (5) and (6) have the property that the amplitude characteristic in the pass-band has monotonic character with maximal number of inflection points. It is worth mentioning that if instead of the coefficients, the polynomial zeroes were used for creation of the characteristic function, one obtains alternative representation of the critical monotonicity as shown in [3] where the characteristic function (for n-odd, for example) is represented as:

$$\frac{\mathrm{d}K(\omega^2)}{\mathrm{d}\omega} = A\cdot\omega\cdot\prod_{i=1}^{(n-1)/2}\left(\omega^2-\omega_i^2\right)^2, \tag{10}$$

with

$$A = 1\bigg/\int_0^1 \omega\cdot\prod_{i=1}^{(n-1)/2}\left(\omega^2-\omega_i^2\right)^2\mathrm{d}\omega. \tag{11}$$

making $K(1) = 1$.

2.1. *The synthesis criteria*

To get a filter function, one is to find the value of the C-constants in (5) and (6). To do that, a design criterion is needed. Four of them were used and may be stated as general. These are:

1. Maximally flat in the origin. This means all derivatives of $L_{\mathrm{n}}(\omega^2)$ at the origin are to be zero. The class of filters thus obtained is called Butterworth's after the author [1]. These will be here referred to as B-filters.
2. Maximum slope of the characteristic function at the edge of the pass-band. The class of filters so obtained is called L-filters and was introduced by Papoupulis [4, 5]. The name L comes from the fact that in the original derivation Legendre polynomials were used. In some references [6], it is stated as 'optimal filters' which is arbitrary.
3. Maximum asymptotic attenuation. This means the higher order coefficient in $L_{\mathrm{n}}(\omega^2)$ has to be maximal. This class of filters was introduced by Halpern [2]. These will be here referred to as H-filters.

Table I. The Jacobi polynomials.

k	$U_k(x)$
0	$\sqrt{2}$
1	$\sqrt{4}\cdot(x)$
2	$\sqrt{6}\cdot(2x^2-1)$
3	$\sqrt{8}\cdot(3x^3-2x)$
4	$\sqrt{10}\cdot(6x^4-6x^2+1)$
5	$\sqrt{12}\cdot(10x^5-12x^3+3x)$
6	$\sqrt{14}\cdot(20x^6-30x^4+12x^2-1)$
7	$\sqrt{16}\cdot(35x^7-60x^5+30x^3-4x)$
8	$\sqrt{18}\cdot(70x^8-140x^6+90x^4-20x^2+1)$
9	$\sqrt{20}\cdot(126x^9-280x^7+210x^5-60x^3+5x)$

DOI: 10.1002/cta

4. Least-squares-monotonic. In this case, the returned power in the pass-band was minimized under the critical monotonicity criterion. This class was introduced by Raković and Litovski [7] and named LSM filters.

Starting with this, some other characteristic functions were produced such as the O-filters [8] where all the *C*-constants were taken to be equal, or the transitional Butterwort-Legendre filters exhibiting properties between the two originals [9–11] and [12]. In [9], the procedure of creating new transfer function was named 'generalization'. Other transitional classes were reported in the literature, e.g. in [13] transition from Butterworth to Halpern filters is described.

In the set [14], and [15], alternative way of derivation of Halpern's results was reported. It was proven in a specific way that this class exhibits maximum attenuation in the infinity. Finally, in [16], the LSM filters were rediscovered after 27 years. Namely, while considering [7] the author claims: 'The coefficients of these filters are obtained by solving a set of nonlinear equations using Newton-Raphson iterative techniques. Such methods are computationally very expensive. In this Letter, a very simple method is presented in which an error function is formulated in a quadratic form and the coefficients are obtained as the eigenvector corresponding to the smallest eigenvalue of a symmetric positive-definite matrix.' After a glimpse to the proposed alternative one can, however, easily conclude that all the steps (with equal complexity) included in Newton-Raphson procedure are involved while, in every iteration, several new are added. Only **C** vector for 10th-order characteristic function was presented (already published in [7]) and no pass-band characteristic of the filter function was given! No new data about the LSM filters was generated. We still think, as we did in 1973, that the Newton-Raphson iteration is the simplest of all. That stands especially in this case since, as will be shown once more in the text below, we are searching for a minimum of a second-order polynomial (parabola) of at most five variables (in the case of the 11th-order filter). Note, once found in [7], **C** became a final not questionable knowledge and there was no need to rediscover it.

In the next, the four major classes will be derived starting with the expressions (5) and (6), i.e. the corresponding values of the *C*-constants will be extracted according to the criteria mentioned. This means the unified presentation will be implemented to synthesis of all classes of monotonic filters.

3. UNIFIED THEORY OF CRITICAL-MONOTONIC FILTERS

We will show in this paragraph that by proper choice of the vector of constants **C**, (5) and (6) may be used to satisfy every criterion as mentioned above. We will derive the **C** vector for all four classes mentioned above. Of course, other criterions may be imposed. The developments here are inspired by the research of Prof. Raković [17] who used a specific approach to show the way of extracting the H, L, and LSM filters by the least-squares criterion. Here, we implement the generic criteria to (5) and (6) in order to create the vector of constants **C** so that a critical-monotonic function is obtained.

a. **Butterworth filters**

To get the values of the *C*- constants for this case, one is to solve the following equations:

$$L_n(\omega^2) = \omega^{2n}. \tag{12}$$

After substitution of (5) and (6) and taking derivatives of both sides, the following systems of linear equations arise

$$\begin{aligned} &\sum_{i=m}^{\frac{n-2}{2}} a_{2i+1,m} C_{2i+1} = 0 \qquad \text{for} \quad m = 0, 1, \quad ...., \frac{n-4}{2} \\ &a_{n-1,(n-2)/2} C_{n-1} = \sqrt{2n} \end{aligned} \tag{13}$$

for *n*-even, and similarly

DOI: 10.1002/cta

$$\sum_{i=m}^{\frac{n-1}{2}} a_{2i,m} C_{2i} = 0 \qquad \text{for} \quad m = 0, 1, \ldots., \frac{n-3}{2} \tag{14}$$
$$a_{n-1,(n-1)/2} C_{n-1} = \sqrt{2n}$$

for n-odd.

Table II contains the solutions of the systems (13) and (14). T2

b. **L- filters**

In this case, we have to implement the criterion of maximum derivative at the end of the pass-band under the constraint expressed by (9).

The derivative of the characteristic function for $\omega = 1$, for n-even, is obtained to be

$$\frac{\mathrm{d}L_n(\omega^2)}{d\omega}_{|\omega=1} = f(C_1, C_3, \ldots, C_{n-1}) = f(\mathbf{C}) = \left\{ \sum_{i=0}^{(n-2)/2} C_{2i+1} U_{2i+1}(1) \right\}^2 \tag{15}$$

where $\mathbf{C}$ is the vector of constants. $\mathbf{C}$ will be found by maximization of (15) under the constraint (9) using the Lagrangian multiplier. It is necessary to maximize the following function:

$$F(\mathbf{C}, \lambda) = f(\mathbf{C}) + \lambda \left\{ \sum_{i=0}^{(n-2)/2} C_{2i+1}^2 - 1 \right\}. \tag{16}$$

After taking the derivatives and equating them to zero, one gets a system of linear equations:

$$\frac{\partial F}{\partial C_1} = 0; \quad \frac{\partial F}{\partial C3} = 0, \ldots, \quad \frac{\partial F}{\partial Cn - 1} = 0, \text{ and } \frac{\partial F}{\partial C_\lambda} = 0 \tag{17}$$

which, after elimination of λ, becomes

$$C_{2i-1} U_{2i+1}(1) - C_{2i+1} U_{2i-1}(1) = 0 \quad \text{for} \quad i = 0, 1, , .., (n-2)/2 \tag{18a}$$

and

$$\sum_{i=0}^{(n-2)/2} C_{2i+1}^2 = 1. \tag{18b}$$

This may be solved recursively to get

$$C_{2i+1} = U_{2i+1}(1) \Big/ \sqrt{\sum_{i=0}^{(n-2)/2} U_{2i+1}^2(1)}. \tag{19}$$

Table II. The C-constants for the maximally flat case.

n	$C_0(C_1)$	$C_2(C_3)$	$C_4(C_5)$	$C_6(C_7)$	$C_8(C_9)$
3	0.866025	0.5			
4	0.942809	0.333333			
5	0.745355	0.645497	0.166666		
6	0.866025	0.489897	0.1		
7	0.661437	0.687386	0.295803	0.05	
8	0.8	0.565685	0.197948	0.028571	
9	0.6	0.692820	0.383325	0.113389	0.014285
10	0.745355	0.602338	0.276641	0.070986	0.007936

DOI: 10.1002/cta

Equivalently, for n-odd, one gets

$$C_{2i} = U_{2i}(1) \Big/ \sqrt{\sum_{i=0}^{(n-1)/2} U_{2i}^2(1)}. \tag{20}$$

Table III contains the numerical values for (19) and (20). T3

3.1. *H-filters*

Here, the criterion is maximum asymptotic attenuation. By inspection of (5) and (6), we easily come to the conclusion that the asymptotic attenuation will be maximal if we choose

$$\begin{aligned} C_{2i} &= 0 \quad \text{for} \quad i = 0, 1, \ldots, (n\text{-}4)/2 \\ C_{n\text{-}1} &= 1 \end{aligned} \tag{21}$$

for n-even, and

$$\begin{aligned} C_{2i+1} &= 0 \qquad \text{for} \qquad i = 0, 1, \ldots, (n\text{-}3)/2 \\ C_{n-1} &= 1 \end{aligned} \tag{22}$$

for n-odd.

Table IV contains the numerical values for (21) and (22). T4

3.2. *LSM filters*

In this case, the criterion is imposed to minimize the area under the characteristic function in the pass-band. The area is obtained by computing squares, hence the name least-squares-monotonic or LSM. In passive filter notation, minimizing that area would mean minimization of the reflected power, hence the physical importance of the criterion.

Table III. The C-constants for the L filters.

n	$C_0(C_1)$	$C_2(C_3)$	$C_4(C_5)$	$C_6(C_7)$	$C_8(C_9)$
3	0.5	0.866025			
4	0.577350	0.816496			
5	0.333333	0.577350	0.745355		
6	0.408248	0.577350	0.707106		
7	0.25	0.433012	0.559016	0.661437	
8	0.316227	0.447213	0.547722	0.632455	
9	0.2	0.346410	0.447213	0.529150	0.6
10	0.258198	0.365148	0.447213	0.516397	0.577350

Table IV. The C-constants for the H filters.

n	$C_0(C_1)$	$C_2(C_3)$	$C_4(C_5)$	$C_6(C_7)$	$C_8(C_9)$
3	0	1			
4	0	1			
5	0	0	1		
6	0	0	1		
7	0	0	0	1	
8	0	0	0	1	
9	0	0	0	0	1
10	0	0	0	0	1

DOI: 10.1002/cta

The area required is obtained by integration:

$$P = \int_0^1 L_n\left(\omega^2\right)\mathrm{d}\omega. \tag{23}$$

This is to be minimal under the condition (9). Again, using Lagrangian multiplier, for n-odd, the following function is obtained:

$$f\left(C_0, C_2, \ldots, C_{n-1}, \lambda\right) = \int_0^1 \left\{ \int_0^{\omega} x \left[\sum_{i=0}^{(n-1)/2} C_{2i} U_{2i}(x) \right]^2 \mathrm{d}x \right\} \mathrm{d}\omega + \lambda \left(\sum_{i=0}^{(n-1)/2} C_{2i}^2 - 1 \right). \tag{24}$$

Similar function may be written for n-even. After proper differentiation and elimination of λ, the following systems of second-order polynomial nonlinear equations were obtained in [7]:

$$\begin{aligned} & C_{2j+1} \cdot \int_0^1 x^2 \left[\sum_{i=0}^{(n-2)/2} C_{2i+1} U_{2i+1}(x) \right] \cdot U_{2j-1}(x) \cdot \mathrm{d}x - \\ & - C_{2j-1} \cdot \int_0^1 x^2 \left[\sum_{i=0}^{(n-2)/2} C_{2i+1} U_{2i+1}(x) \right] \cdot U_{2j+1}(x) \cdot \mathrm{d}x = 0 \qquad \text{for } j = 1, 2, \ldots, (n\text{-}2)/2 \\ & \sum_{i=0}^{(n-2)/2} C_{2i+1}^2 = 1 \end{aligned} \tag{25}$$

for n-even, and

$$\begin{aligned} & C_{2j} \cdot \int_0^1 x^2 \left[\sum_{i=0}^{(n-1)/2} C_{2i} U_{2i}(x) \right] \cdot U_{2j-2}(x) \cdot \mathrm{d}x - \\ & - C_{2j-2} \cdot \int_0^1 x^2 \left[\sum_{i=0}^{(n-1)/2} C_{2i} U_{2i}(x) \right] \cdot U_{2j}(x) \cdot \mathrm{d}x = 0 \qquad \text{for } j = 1, 2, \ldots, (n\text{-}1)/2 \\ & \sum_{i=0}^{(n-1)/2} C_{2i}^2 = 1 \end{aligned} \tag{26}$$

for n-odd.

These were solved by Newton-Raphson iteration and the results are given in Table V. T5

4. COMPARISON OF THE PROPERTIES OF THE BASIC CLASSES OF CRITICAL CRITICAL-MONOTONIC FILTERS

In this paragraph, numerical values will be given enabling comparisons of the four classes of monotonic filters derived above.

To get an intuitive feeling about the properties of the four classes of filters described above, in Figure 1, F1 the amplitude characteristic is presented. In fact, $L_n(\omega^2)$ is given in logarithmic scale. One gets the general feeling that the LSM filters are the best approximant in the pass-band while exhibiting good selectivity in the stop-band. As expected, the H characteristic has the highest attenuation in the stop-band while the L filters may be considered as a kind of compromise between the LSM and the H filters. Except for the maximally flat property, no advantage can be seen for the Butterworth's filters.

For quantitative comparison, the values of the quantities related to the three main criteria will be listed below for all four classes of filters.

The area P, calculated from (23), will be considered first. Proper values are given in Table VI. T6

DOI: 10.1002/cta

Table V. The C-constants for the LSM filters.

n	$C_0(C_1)$	$C_2(C_3)$	$C_4(C_5)$	$C_6(C_7)$	$C_8(C_9)$
3	0.735595	0.677422			
4	0.816497	0.577350			
5	0.539066	0.716797	0.442277		
6	0.645810	0.661605	0.381962		
7	0.425843	0.629268	0.570416	0.311037	
8	0.529322	0.615107	0.516024	0.274191	
9	0.349642	0.547621	0.565019	0.451636	0.233753
10	0.446303	0.553681	0.531873	0.409607	0.208768

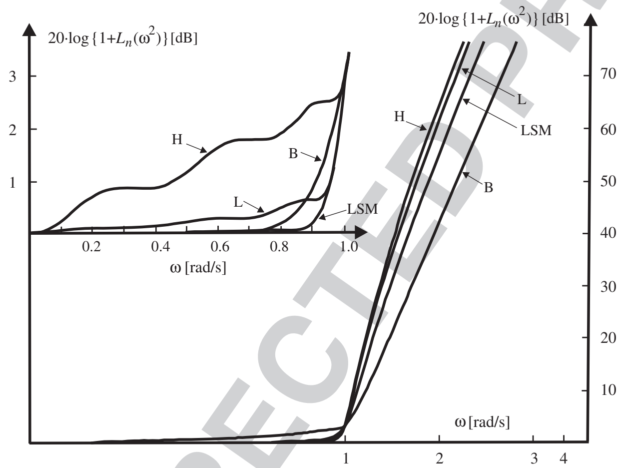


Figure 1. The four main critical-monotonic approximants for $n = 7$.

Table VI. Area under the $L_n(\omega^2)$ curve.

n Type	3	4	5	6	7	8	9	10
B	0.1429	0.1111	0.0999	0.0769	0.0667	0.0588	0.0526	0.0476
H	0.3714	0.3143	0.3679	0.3390	0.3663	0.348	0.3654	0.3534
L	0.1619	0.1238	0.1071	0.0894	0.0800	0.0700	0.0640	0.0574
LSM	0.121	0.086	0.061	0.047	0.037	0.030	0.025	0.021

Note that for the 10th-order LSM filter, only approximately 2% of the area (i.e. energy of the signal spectrum in the pass-band) is 'wasted'. The next best approximation exhibits more than twice larger deviation from ideality. Note that from this point of view, the H filters, which are the worst solution, are getting no better when the order of the filter is raised.

The slope of $L_n(\omega_2)$ for $\omega = 1$ is given in Table VII. T7

By inspection of Table VII, we come to the conclusion that the maximum slope at the edge of the pass-band exhibited by the L-filters is followed by the LSM filters. In the case of the tenth order filter, the ratio of the slopes is maximal, being 1.27 in favour to the L filters. For lower orders, this difference is diminishing.

Finally, the asymptotic attenuation will be expressed as value of the coefficient in $L_n(\omega^2)$ multiplying ω^{2n}. The results are shown in Table VIII. T8

To get a picture about the mutual relations of the approximants, the quotients (expressed in dB) of the values in the last column of Table IX, with the value for the H filter as the denominator, will be computed. So, for the 10th order, for the H-versus-B filter, we have: $20 \cdot \log(126/1) = 42$ [dB], for the H-versus-L filter, we have: $20 \cdot \log(126/36.7) = 10.71$ [dB], and for the H-versus-LSM filter, we have: $20 \cdot \log(126/26.3) = 13.61$ [dB]. These are representing the difference in the attenuation at infinity. T9

DOI: 10.1002/cta

Table VII. The slope of $L_n(\omega^2)$ at $\omega = 1$.

n Type	3	4	5	6	7	8	9	10
B	6	8	10	12	14	16	18	20
H	6	8	10	12	14	16	18	20
L	8	12	18	24	32	40	50	60
LSM	7.288	10.67	15.34	20.10	26.16	32.29	39.74	47.24

Table VIII. The value of $\lim\limits_{\omega \to \infty} [L_n(\omega^2)/\omega^{2n}]$.

n Type	3	4	5	6	7	8	9	10
B	1	1	1	1	1	1	1	1
H	2	3	6	10	20	35	70	126
L	1.41	2.45	4.47	7.07	13.23	22.14	42	36.37
LSM	1.35	1.73	2.65	3.82	6.22	9.60	16.36	26.30

Table IX. The coefficients of the denominator polynomials of the B filters.

s^{10}	s^9	s^8	s^7	s^6	s^5	s^4	s^3	s^2	s^1	s^0
									1.414214	1.
							1.	2.000000	2.000000	1.
						1.	2.613126	3.414214	2.613126	1.
					1.	3.236068	5.236068	5.236068	3.236068	1.
				1.	3.863703	7.464102	9.141620	7.464102	3.863703	1.
			1.	4.493959	10.097835	14.591794	14.591794	10.097835	4.493959	1.
		1.	5.125831	13.137071	21.846151	25.688356	21.846151	13.137071	5.125831	1.
	1.	5.758770	16.581719	31.163437	41.986386	41.986386	31.163437	16.581719	5.758770	1.
1.	6.392453	20.431729	42.802061	64.882396	74.233429	64.882396	42.802061	20.431729	6.392453	1.

Note that LSM filters are for less than 3 dB worse than the L filters. That difference is smaller for lower orders of the filters.

As a general conclusion, we may state that B filters being derived to minimize distortions in the pass-band are not the best solution for that purpose. Namely, use of LSM filter will provide lower distortions in the pass-band while exhibiting higher selectivity, i.e. narrower transition region. Similarly, the L filter derived to have maximal slope at the band-edge exhibits no narrower transition region. The H filters are by no doubt the most selective critical-monotonic filters. This suggests that the trade-off between pass-band distortions and stop-band attenuation is to be sought between the LSM and H filters.

5. STATE-VARIABLE IMPLEMENTATION OF THE CRITICAL-MONOTONIC ALL-POLE FILTERS

There are several options for implementation of a given transfer function as an RC active continuous time filter. Among others here, for the implementation of the critical-monotonic all-pole filters, we advise the use of the so-called state-variable filter structure. As an example, a schematic realizing a fifth-order low-pass filter is depicted in Figure 2. F2

This structure attracts attention for a long period of time, e.g. [18, 19]. The main advantages of this structure may be listed as follows:

- Exhibits the simplest of all relation between the element values and the filter's transfer function coefficients
- Maintains separate adjustment of each operating coefficient,

DOI: 10.1002/cta

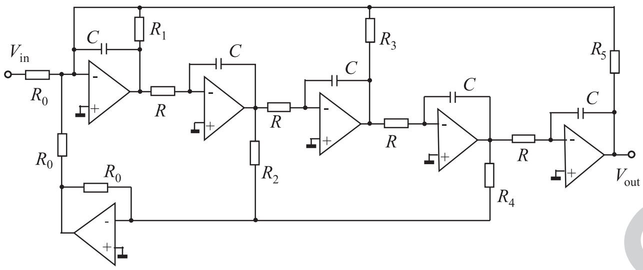


Figure 2. Fifth-order all-pole low-pass state-variable filter.

- Allows gain tuning of all operating coefficients,
- Multiple inputs and/or outputs are possible.

A special property of these structures, if compared with cascaded realization of second- and/or third-order cells, is that now no attention is to be paid to the order of extraction of the poles, i.e. the order of the cells in the cascade. Namely, since every cell has different nominal gain, if the one with highest gain is set at the input, one risks to produce distortions of the input signal. In the opposite case, when the one with smallest gain is set as the input cell of the filter, one generally gives rise to the noise. Accordingly, an algorithm for optimal ordering of the cells in the cascade would be necessary to be implemented in that case.

The transfer function of the state-variable filter of nth order is given by

$$T(s) = \frac{RG_0}{\sum_{i=0}^{n} RG_{n-i}(sCR)^i} \tag{27}$$

where $G_i = 1/R_i$, for $i = 0,1,2,\ldots,n-1$, and $G_n = 1/R$.

So if the normalization frequency $\omega_{norm} = 1/(RC)$ is used, the coefficients of the denominator polynomial are numerically equal to the normalized conductance values in Figure 2. This dramatically simplifies the implementation of a state-variable filter based on data given in the tables given below. This is why in the next we are listing the coefficients of the denominator polynomials of the four basic classes of critical-monotonic all-pole filters discussed above (Tables X–XII). Q1 **T10 – T12**

6. DESIGN EXAMPLE

To design an all-pole filter with critical-monotonic amplitude in a form of a state-variable filter, we have to make two decisions: choice of the filter type (among the four classes) and choice of the filter order n. To do that, we need two additional data. For the choice of the class of filters, we have to think on the requirements in the following way. First, looking to the attenuation characteristic in the stop-band, one would always prefer the H filters. When looking to the pass-band, however, additional arguments come in fore mainly related to the shape of the spectrum of the signal which is processed. In most cases, the energy of the

Table X. The coefficients of the denominator polynomials of the H filters.

s^{10}	s^9	s^8	s^7	s^6	s^5	s^4	s^3	s^2	s^1	s^0
							1.0000	1.1070	1.3627	0.4999998
						1.0000	1.3084	1.7449	1.0785	0.3333335
					1.0000	1.2253	2.0007	1.3535	0.7681405	0.1666665
				1.0000	1.3607	2.3658	1.9082	1.3280	0.5153700	0.1000007
			1.0000	1.3169	2.6171	2.1957	1.8819	0.8954657	0.3271794	0.0499999
		1.0000	1.4187	2.9656	2.7668	2.6300	1.5218	0.7208994	0.2029639	0.0285715
	1.0000	1.3896	3.2155	3.0712	3.3225	2.1019	1.2390	0.4517988	0.1214301	0.0142857
1.	1.4512	3.5253	3.6016	4.1933	2.9624	1.9859	0.8937465	0.3222619	0.0712828	0.0078896

DOI: 10.1002/cta

Table XI. The coefficients of the denominator polynomials of the L filters.

s^{10}	s^9	s^8	s^7	s^6	s^5	s^4	s^3	s^2	s^1	s^0
							1.0000	1.3107	1.3590	0.577.3484
						1.0000	1.5629	1.8879	1.2416	0.4082493
					1.0000	1.5515	2.2036	1.6927	0.898.3409	0.2236066
				1.0000	1.7261	2.6897	2.4334	1.6331	0.6796352	0.1414213
			1.0000	1.7279	2.9928	2.9246	2.3322	1.2308	0.4379427	0.075.5929
		1.0000	1.8614	3.4466	3.7232	3.3477	2.1189	0.9939308	0.2996703	0.0451753
	1.0000	1.8663	3.7416	4.2490	4.2477	3.0119	1.7074	0.6804341	0.1815714	0.0238093
1.	1.4443	3.2653	3.5424	3.9521	3.0847	2.1108	1.0830	0.4220421	0.1077180	0.0137465

Table XII. The coefficients of the denominator polynomials of the LSM filters.

s^{10}	s^9	s^8	s^7	s^6	s^5	s^4	s^3	s^2	s^1	s^0
							1.0000	1.6112	1.5770	0.7386759
						1.0000	1.9449	2.3357	1.6423	0.5773505
					1.0000	2.0904	2.9119	2.5154	1.3811	0.3768382
				1.0000	2.3347	3.6550	3.7126	2.6098	1.1704	0.2624239
			1.0000	2.4426	4.1923	4.7414	3.9215	2.2696	0.8536734	0.1602889
		1.0000	2.6335	4.8946	6.1169	5.7487	3.9801	1.9802	0.6424145	0.1042032
	1.0000	2.7183	5.3970	7.2412	7.5049	5.8983	3.5225	1.5214	0.4313536	0.0611156
1.	2.8760	6.0631	8.7367	9.8614	8.6259	5.9248	3.1115	1.1937	0.3012679	0.0380161

signal is located in the lower part of the pass-band and implementation of LSM amplitude characteristic is preferable. If LSM is chosen, one should expect higher order of the filter to be implemented (in comparison to the H filters) for the given stop-band requirements to be satisfied.

In the example, here we chose the filter attenuation characteristic defined in Figure 3. There, we ask for the attenuation in the stop-band to reach 60 dB (1000 times) at the frequency 2.5 times higher than the cut-off frequency of the filter being $f_c = 3.2$ kHz. **F3**

To make a choice according to the consideration mentioned, one should create quantitative information about the dependence of the stop-band attenuation for a given frequency on the order of the filter. That is done for $n = 6$ in Figure 4. There, on the abscissa, the attenuation is given while the ordinate is representing the corresponding normalized frequency for the proper filter, i.e. the stop-band attenuation characteristic is inverted. We can read that for $n = 6$, at the frequency $\omega = 2.5 \cdot \omega_c$, the H filter exhibits 65+ [dB]. Similar attenuation values may be seen for the L filter. The LSM filter, however, exhibits only 57+ [dB] meaning that the H filter and L filter satisfy the stop-band requirements with $n = 6$, while the LSM solution will need a seventh order filter. So, we see that a trade-off between the pass-band magnitude distortion and the stop-band performance is possible. If the pass-band requirements are prevailing, however, one will use the LSM solution paying the price of **F4**

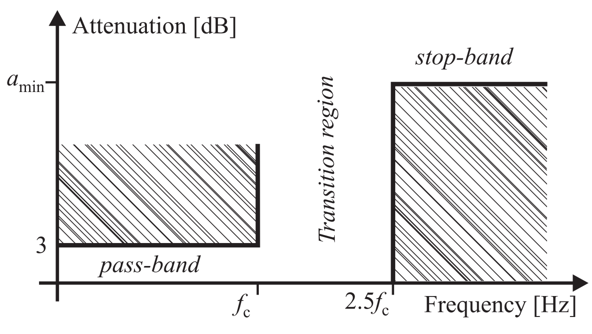


Figure 3. Definition of the filter requirements. Here, $f_c = 3.2$ kHz and $a_{min} = 60$ dB.

DOI: 10.1002/cta

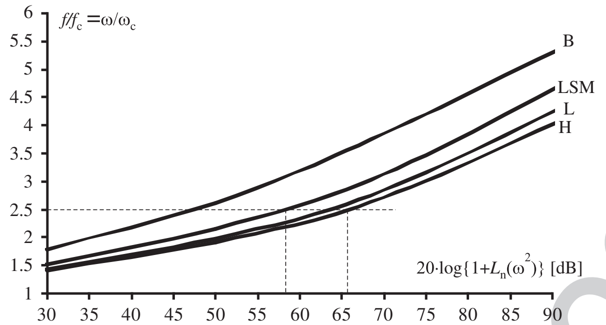


Figure 4. Attenuation characteristics of sixth-order filters in the stop-band.

one operational amplifier, two resistors, and one capacitor needed for realization of the state-variable version of the active filter.

So, for the requirements expressed in Figure 3, we have $RC = (1/\omega_{\mathrm{norm}}) = (1/\omega_c) = [1/(2\pi \cdot 3200)] = 4.9736 \cdot 10^{-5}$ s. If we choose $R = 10$ kΩ, we get $C = 4.9736$ nF. The rest of the resistances are easily calculated from

$$\sum_{i=0}^{n} a_i s_{\mathrm{n}}^i = \sum_{i=0}^{n} RG_{n-i}(sCR)^i \tag{28}$$

where **a** is the vector of coefficients in the denominator of the transfer function and $s_{\mathrm{n}} = sRC$ is the normalized complex frequency. By comparison, we get: $RG_{n-i} = a_i$ or $R_{n-i} = R/a_i$.

If the seventh-order LSM and the sixth-order H solutions are taken, the values given in Table XIII arise. **T13**

Note the same state-variable circuit synthesis procedure may be applied to any all-pole filter.

The amplitude characteristics of the seventh-order LSM and the sixth-order H state-variable filters obtained by SPICE simulation are depicted in Figure 5. The operational amplifier was modelled as **F5** ideal with a gain of 10^5.

When comparing filter functions, some additional information is frequently needed. Namely, the mapping of the element tolerances into the response variations may be of interest and help the selection of the solution. These data may lead to the estimation of the yield in series production and because of that may be of decisive importance. To get the solution properties from the tolerance point of view, Monte Carlo simulation of the seventh-order LSM, and the sixth-order H state-

Table XIII. Resistances within the filters (Ω).

	R_7	R_6	R_5	R_4	R_3	R_2	R_1	R_0
Seventh-order LSM	62387.35	11714.08	4406.06	2550.04	2109.08	2385.33	4094.00	10000.0
Sixth-order H		100000.	19403.5	7530.12	5240.54	4226.90	7349.16	10000.0

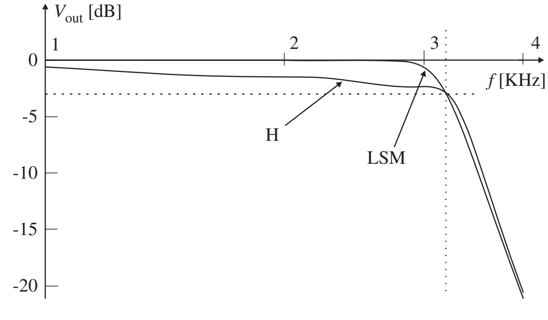


Figure 5. Amplitude characteristics of the seventh-order LSM and sixth-order H filters.

DOI: 10.1002/cta

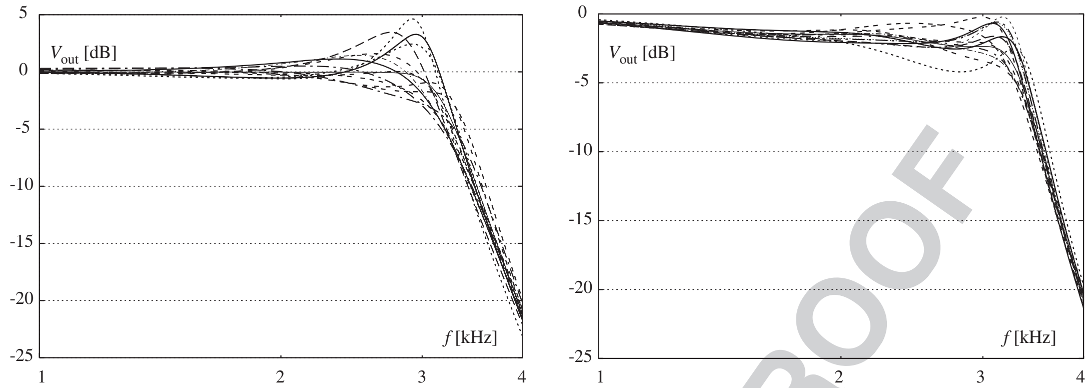


Figure 6. Monte-Carlo simulation of the seventh-order LSM (left) and sixth-order H (right) state-variable filter (part of the amplitude–frequency characteristic).

variable filters were performed with all element values having 1% standard deviation and Gaussian distribution. The simulation results are depicted in Figure 6. Here part of the amplitude characteristic **F6** around the cut-off frequency is given for the interval where the sensitivity is supposed to be the highest. As expected, the LSM filter exhibits very low variations in the main part of the pass-band. At the band-edge, however, due to the higher skirt slope of the amplitude characteristic, it produces larger variations than the H filter.

When considering the last results, one should have in mind that the tolerance mapping is governed by two factors. First is the very transfer function as a mathematical expression and second is the physical realization in a form of electrical or electronic circuit. Having that in mind, one may claim that for a different realization (e.g. passive or cascaded active), different mappings will be obtained. In general, however, one may expect some similarity to the ones depicted in Figure 6.

7. CONCLUSION

The subject of synthesis of critical-monotonic low-pass amplitude characteristics was revisited. Several new contributions were given in order to: facilitate the choice of the proper transfer function, to allow cataloguing the transfer functions, to simplify the circuit synthesis procedure, and to perform synthesis in the form of a state-variable continuous time active filter. In that way, we hope, a platform is established for choice of more appropriate monotonic solutions depending on the design requirements imposed. Four main criteria for transfer function synthesis were implemented: maximally flat at the origin, maximum slope at the band-edge, maximal asymptotic attenuation, and minimal amplitude distortion in the pass-band. For every criterion, a class of filters was generated and the coefficients of the transfer functions were generated and published for the first time (with one exception). Properties of the classes so generated were quantitatively compared for the first time. The state-variable structure was advised as the one with the simplest synthesis procedure. A procedure was explained and the design process was exemplified. Statistical tolerance analysis was performed for the example solutions in order to complete the picture for comparison.

ACKNOWLEDGEMENTS

This research was partially funded by The Ministry of Education and Science of Republic of Serbia under contract No. TR32004.

REFERENCES

1. Butterworth S. On the theory of filter amplifiers. *Experimental Wireless and the Wireless Engineer* 1930; **7**:536–541.
2. Halpern P. Optimum monotonic lowpass filters. *IEEE Transaction on a Circuit Theory* 1969;240–241. Q2

DOI: 10.1002/cta

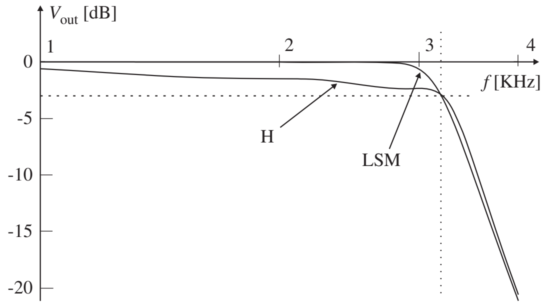
3. Rabrenović D, Jovanović V. Low-pass filters with critical monotonic magnitude. *Publications of Faculty of Electrical Engineering, ETA series* 1973; Belgrade, 59–68.
4. Papoulis A. Optimum filters with monotonic response. *Proceedings of the IRE* 1958; (**46**):606–609.
5. Papoulis A. On monotonic response filters. *Proceedings of the IRE* 1959; (**47**):332–333.
6. Fukada M. Optimum filters of even orders with monotonic response. *IRE Transaction on a Circuit Theory* 1959; 277–281.
7. Raković BD, Litovski VB. Least-squares monotonic low-pass filters with sharp cutoff. *Electronics Letters* 1973; (**9**):75–76.
8. Raković BD, Lazović SM. Monotonic low-pass filters with improved stop-band performance. *IEEE Transaction on a Circuit Theory* 1972; (**CT-19**):218–221.
9. Filanovsky IM. A Generalization of Filters with Monotonic Amplitude-Frequency Response. *IEEE Transaction on Circuits and Systems Part I: Fundamental Theory and Applications* 1999; **46**(11):1382–1385.
10. Rakovich BB. Transitional Butterworth-Legendre Filters. *The Radio and Electronic Engineer* 1974; **44**(12):673–680.
11. Filanovsky IM. Filters with Monotonic Amplitude-Frequency Response. *Proceedings Midwest Symposium on Circuits and Systems* 1998; 436–439.
12. Ueda T, Aikawa N, Sato M. Design method of analog lowpass filters with monotonic response and arbitrary flatness. *IEEE International Conference on Circuits and Systems* 1998; (**3**):15–18.
13. Djurich BM, Petkovich RA. Generalized Analysis of Optimum Monotonic Low-Pass Filters. *IEEE Transactions on Circuits and Systems* 1976; **CAS-23**(11):647–649.
14. Djurich BM. Monotonic low-pass filters with maximum selectivity. *Electronics Letters* 1975; **11**(4):82–83.
15. Phuc DT. Comment: Monotonic lowpass filters with maximum selectivity. *Electronics Letters* 1978; **14**(12):362–364. Reply: Djurich, B. M., p. 364.
16. Kidambi SS. Simple method for design of monotonic analogue filters. *Electronics Letters* 2000; **36**(4):287–288.
17. Rakovich BD. Designing monotonic low-pass filters-comparison of some methods and criteria. *International Journal of Circuit Theory and Applications* 1974; **2**(3):215–221.
18. Scanlan SO. Analysis and Synthesis of Switched-Capacitor State-Variable Filters. *IEEE Transaction on Circuits and Systems* 1981; **CAS-28**(2):85–93.
19. Pandit S, Kar S, Mandal C, Patra A. High Level Synthesis of Higher Order Continuous Time State Variable Filters with Minimum Sensitivity and Hardware Count. *Proceedings of the Design, Automation, and Test in Europe, DATE '06* 2006; (1):1–2.

DOI: 10.1002/cta

Research Article

Unified theory and state-variable implementation of critical-monotonic all-pole filters

Dragan Topisirović, Vančo Litovski and Miona Andrejević Stošović



Four criteria are implemented to synthesize, in a unified manner, the transfer functions of all-pole monotonic amplitude filters. Comparisons of the properties of the filter classes and coefficients of the transfer functions are published the first time. The state-variable structure is advised as the one with the simplest synthesis procedure which is exemplified. Statistical tolerance analysis is performed for the example solutions in order to complete the information needed for comparisons. The figure depicts comparison of two alternatives.

Author Query Form

Journal: International Journal of Circuit Theory and Applications

Article: cta_1956

Dear Author,

During the copyediting of your paper, the following queries arose. Please respond to these by annotating your proofs with the necessary changes/additions.

- If you intend to annotate your proof electronically, please refer to the E-annotation guidelines.
- If you intend to annotate your proof by means of hard-copy mark-up, please refer to the proof mark-up symbols guidelines. If manually writing corrections on your proof and returning it by fax, do not write too close to the edge of the paper. Please remember that illegible mark-ups may delay publication.

Whether you opt for hard-copy or electronic annotation of your proofs, we recommend that you provide additional clarification of answers to queries by entering your answers on the query sheet, in addition to the text mark-up.

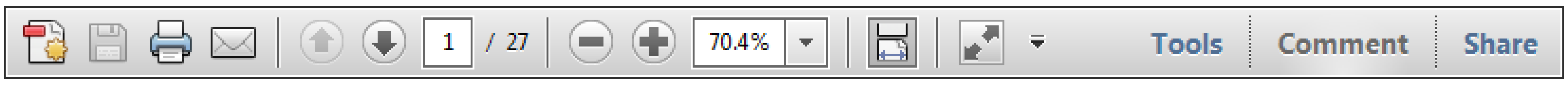
Query No.	Query	Remark
Q1	AUTHOR: Tables X–XII were not cited in the text. An attempt has been made to insert the tables into a relevant point in the text – please check that this is OK. If not, please provide clear guidance on where they should be cited in the text.	
Q2	AUTHOR: Please supply vol. no. of refs. 2, 6 and 11.	

WILEY-BLACKWELL

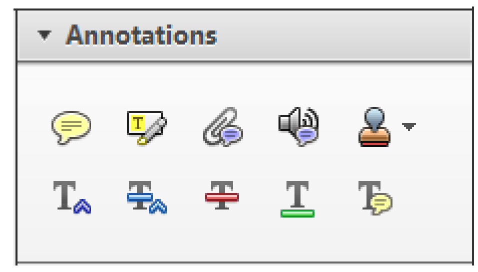
USING e-ANNOTATION TOOLS FOR ELECTRONIC PROOF CORRECTION

Required software to e-Annotate PDFs: Adobe Acrobat Professional or Adobe Reader (version 7.0 or above). (Note that this document uses screenshots from Adobe Reader X)
The latest version of Acrobat Reader can be downloaded for free at: http://get.adobe.com/uk/reader/

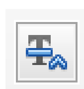
Once you have Acrobat Reader open on your computer, click on the Comment tab at the right of the toolbar:



This will open up a panel down the right side of the document. The majority of tools you will use for annotating your proof will be in the Annotations section, pictured opposite. We've picked out some of these tools below:



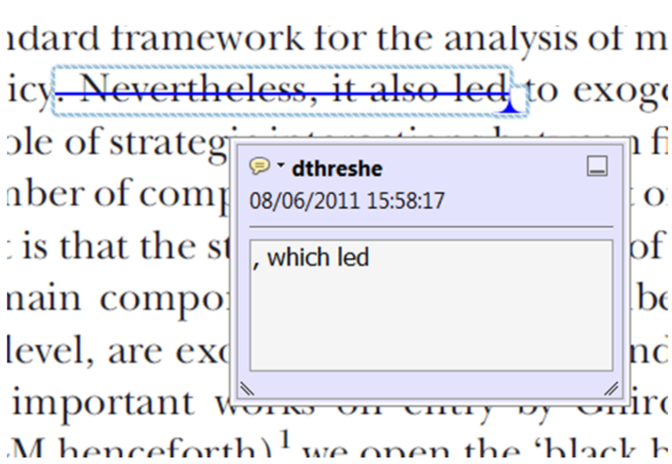
1. Replace (Ins) Tool – for replacing text.



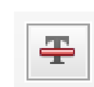
Strikes a line through text and opens up a text box where replacement text can be entered.

How to use it

- Highlight a word or sentence.
- Click on the Replace (Ins) icon in the Annotations section.
- Type the replacement text into the blue box that appears.



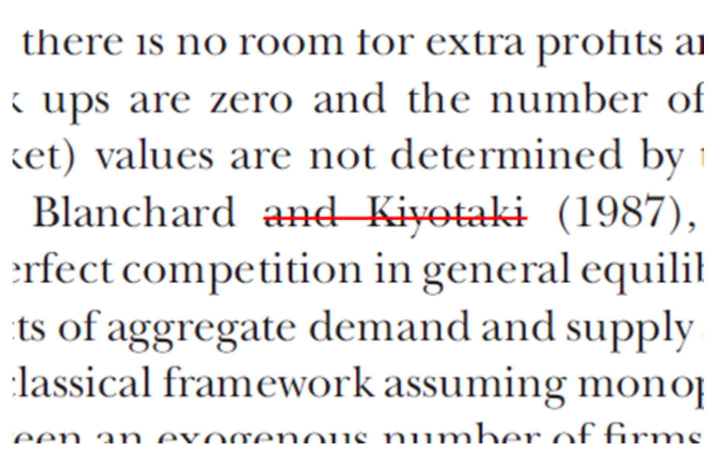
2. Strikethrough (Del) Tool – for deleting text.



Strikes a red line through text that is to be deleted.

How to use it

- Highlight a word or sentence.
- Click on the Strikethrough (Del) icon in the Annotations section.

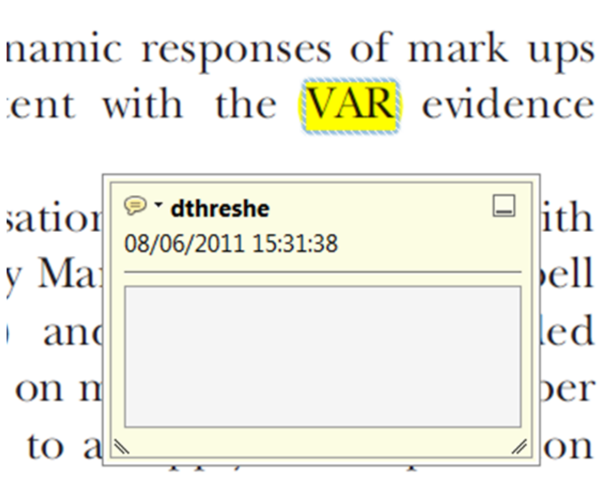


3. Add note to text Tool – for highlighting a section to be changed to bold or italic.

Highlights text in yellow and opens up a text box where comments can be entered.

How to use it

- Highlight the relevant section of text.
- Click on the Add note to text icon in the Annotations section.
- Type instruction on what should be changed regarding the text into the yellow box that appears.

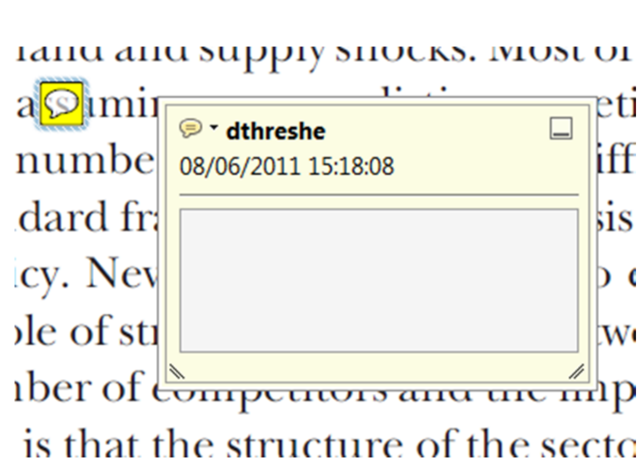


4. Add sticky note Tool – for making notes at specific points in the text.

Marks a point in the proof where a comment needs to be highlighted.

How to use it

- Click on the Add sticky note icon in the Annotations section.
- Click at the point in the proof where the comment should be inserted.
- Type the comment into the yellow box that appears.



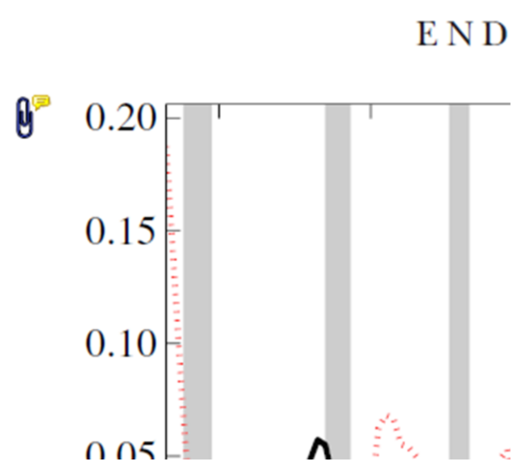
USING e-ANNOTATION TOOLS FOR ELECTRONIC PROOF CORRECTION

5. Attach File Tool – for inserting large amounts of text or replacement figures.

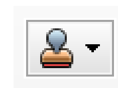
Inserts an icon linking to the attached file in the appropriate pace in the text.

How to use it

- Click on the Attach File icon in the Annotations section.
- Click on the proof to where you'd like the attached file to be linked.
- Select the file to be attached from your computer or network.
- Select the colour and type of icon that will appear in the proof. Click OK.



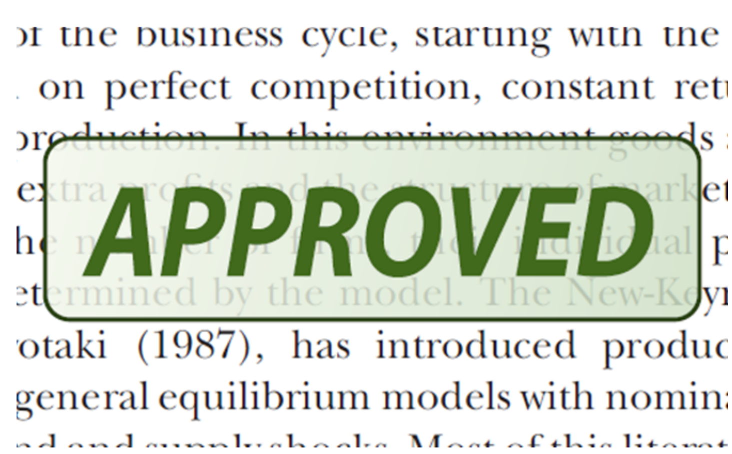
6. Add stamp Tool – for approving a proof if no corrections are required.

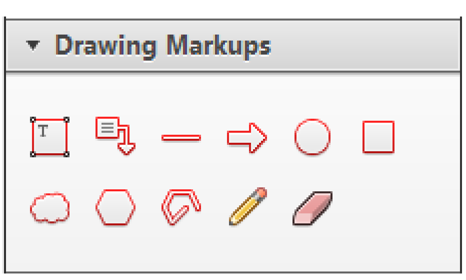


Inserts a selected stamp onto an appropriate place in the proof.

How to use it

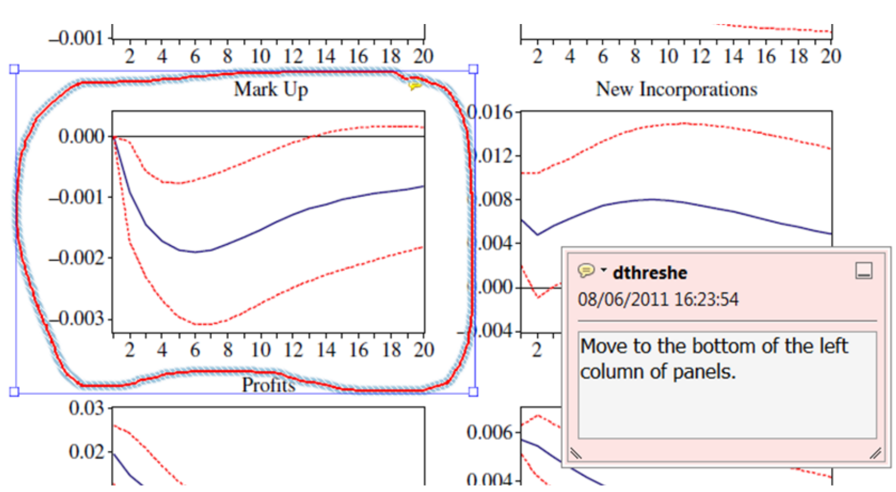
- Click on the Add stamp icon in the Annotations section.
- Select the stamp you want to use. (The Approved stamp is usually available directly in the menu that appears).
- Click on the proof where you'd like the stamp to appear. (Where a proof is to be approved as it is, this would normally be on the first page).





7. Drawing Markups Tools – for drawing shapes, lines and freeform annotations on proofs and commenting on these marks.

Allows shapes, lines and freeform annotations to be drawn on proofs and for comment to be made on these marks..



How to use it

- Click on one of the shapes in the Drawing Markups section.
- Click on the proof at the relevant point and draw the selected shape with the cursor.
- To add a comment to the drawn shape, move the cursor over the shape until an arrowhead appears.
- Double click on the shape and type any text in the red box that appears.

For further information on how to annotate proofs, click on the Help menu to reveal a list of further options: