ON TWO EFFICIENT ROOT-SOLVERS WITH MEMORY

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THE AIM OF COMMUNICATION

- **The aim of this communication is the construction, efficiency analysis and numerical testing of two new one-step methods with memory for solving nonlinear equations of the form $f(x) = 0$, where f is a function of a real or complex variable.**

- **The presented methods posses very high efficiency and can be used as predictor steps in constructing multipoint methods.**

ONE-STEP METHODS

Newton method (N)**:** $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$,

Steffensen method: $x_{k+1} = x_k - \dfrac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}$,

Traub-Steffensen method $(T - S)$**:**

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)},$$

Halley methos: $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k) - \dfrac{f''(x_k) f(x_k)}{2 f'(x_k)}}$.

ONE-STEP METHODS WITH MEMORY (EARLY WORKS)

Traub-Steffensen's method with memory $(T-S\ M)$ **:**

$$\begin{cases} \gamma_0 \text{ is given}, \quad \gamma_k = -\dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \text{ for } k \geq 1, \\ x_{k+1} = x_k - \dfrac{\gamma_k f(x_k)^2}{f(x_k + \gamma_k f(x_k)) - f(x_k)}, \qquad (k = 0, 1, \ldots) \end{cases}$$

with the order of convergence at least $1+\sqrt{2} \approx 2.414.$

ONE-STEP METHODS WITH MEMORY (EARLY WORKS)

(CONTINUATION)

Traub's method with memory $(T\ M)$ **of order no less than** $1+\sqrt{3} \approx 2.732$

$$\begin{cases} p_0 \text{ is given}, \quad p_k = -\dfrac{H_3''(x_k)}{2f'(x_k)} \text{ for } k \geq 1, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k) + p_k f(x_k)}, \qquad (k = 0, 1, \ldots), \end{cases}$$

where

$$H_3''(x_k) = \frac{2}{x_k - x_{k-1}} \left(2f'(x_k) + f'(x_{k-1}) - 3f[x_k, x_{k-1}]\right).$$

RECENT RESULTS

M. S. Petković J. Džunić, L. D. Petković, A family of two-point methods with memory for solving nonlinear equations, Appl. Anal. Discrete Math. 5 (2011), 298–317.

J. Džunić, M. S. Petković, L. D. Petković, Three-point methods with and without memory for solving nonlinear equations, Appl. Math. Comput. 218 (2012), 4917–4927.

J. Džunić, M. S. Petković, On generalized multipoint root-solvers with memory, J. Comput. Appl. Math. 236 (2012) 2909–2920.

MODIFIED STEFFENSEN METHOD

Let α be a simple real zero of a real function $f: D \subset R \to R$ and let x_0 be an initial approximation to α, and $p \in R$ a parameter. Steffensen method modification:

$$\begin{cases} w_k = x_k + \gamma f(x_k), \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f[w_k, x_k] + pf(w_k)} \quad (k = 0, 1, \ldots), \end{cases} \tag{1}$$

Introduce notation $c_2 = f''(\alpha)/(2f'(\alpha))$, $\varepsilon_k = x_k - \alpha$ and $\varepsilon_{k,w} = w_k - \alpha$. Error relation of the method (1):

$$\varepsilon_{k+1} \sim (1 + \gamma f'(\alpha))(c_2 + p)\varepsilon_k^2,$$

plays the key role in convergence acceleration.

ACCELERATION OF THE MODIFIED STEFFENSEN METHOD

Minimizing of terms $1+\gamma f'(\alpha)$ and c_2+p is obtained by the following models:

Model (I)

$$\begin{cases} \gamma_k = -\dfrac{1}{f[x_k, w_{k-1}]} = -\dfrac{1}{N_1'(x_k)}, \\ p_k = -\dfrac{f[w_k, x_k, w_{k-1}]}{f[w_k, x_k]} = -\dfrac{N_2''(w_k)}{2f[w_k, x_k]}, \end{cases} \tag{2}$$

where

$$N_1(\tau) = N_1(\tau; x_k, w_{k-1}) \quad \text{and} \quad N_2(\tau) = N_2(\tau; w_k, x_k, w_{k-1});$$

ACCELERATION OF THE MODIFIED STEFFENSEN METHOD

(CONTINUATION)

Model (II)

$$\begin{cases} \gamma_k = -\dfrac{1}{N_2'(x_k)}, \\ p_k = -\dfrac{N_3''(w_k)}{2f[w_k, x_k]}, \end{cases} \tag{3}$$

where

$$N_2(\tau) = N_2(\tau; x_k, w_{k-1}, x_{k-1}) \text{and}$$

$$N_3(\tau) = N_3(\tau; w_k, x_k, w_{k-1}, x_{k-1});$$

NEW METHOD WITH MEMORY OF STEFFENSEN TYPE

After substituting constant parameters γ and p by γ_k and p_k in each iteration step, we obtain

$$\begin{cases} \gamma_0, \ p_0 \text{ are given}, \\ \gamma_k = -\dfrac{1}{N'_m(x_k)}, \quad p_k = -\dfrac{N''_{m+1}(w_k)}{2f[w_k, x_k]}, \quad (m = 1, 2) \text{ for } k \geq 1, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f[w_k, x_k] + p_k f(w_k)} \quad (k = 0, 1, \ldots). \end{cases} \tag{4}$$

CONVERGENCE THEOREM I

Theorem 1. *If an initial approximation* x_0 *is sufficiently close to a zero* α *of* f, *then the order of convergence of the Steffensen-like method with memory* (4) *is at least three if* γ_k *and* p_k *are calculated by* (2)*. If formulae* (3) *are used for calculating* γ_k *and* p_k, *then the order* $\frac{1}{2}(3+\sqrt{17}) \approx 3.56$ *is achieved.*

MODIFIED NEWTON METHOD

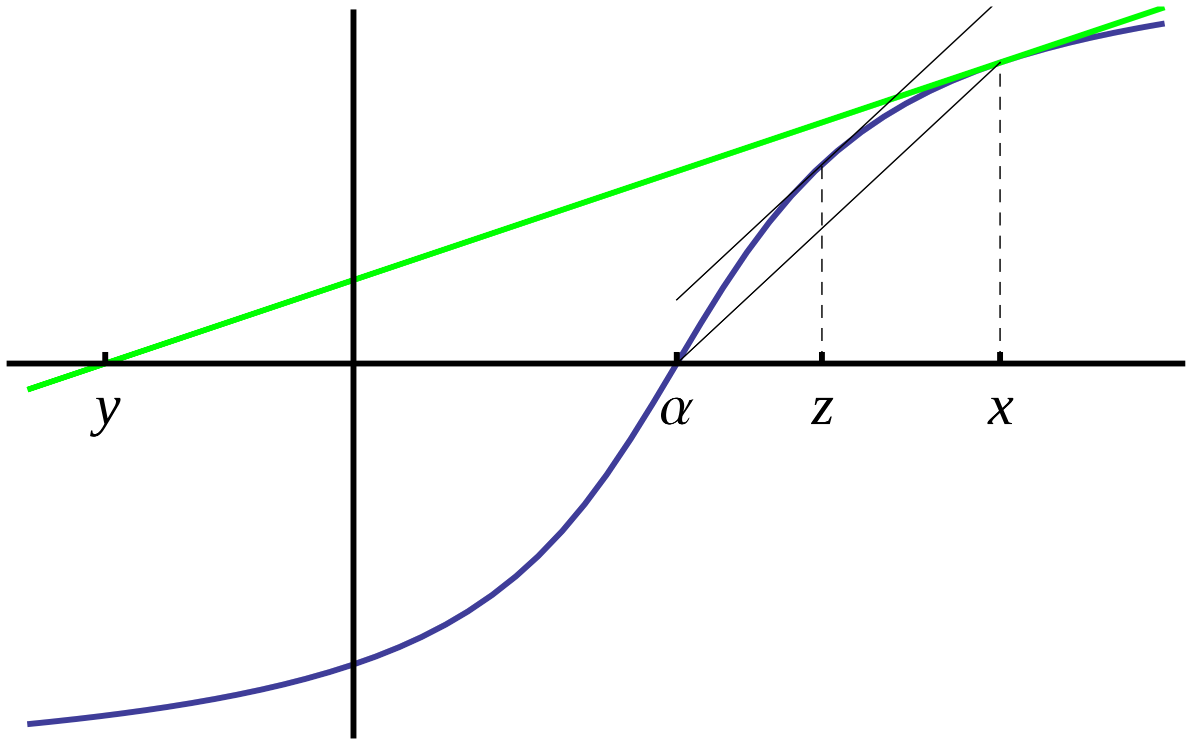

FIGURE 1 **Geometric interpretation for the new method**

MODIFIED NEWTON METHOD

New approximation for the sought simple zero α calculated by

$$\begin{cases} w_k = x_k + \gamma f(x_k), \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(w_k)} \end{cases} \tag{5}$$

has an error relation of the form

$$\varepsilon_{k+1} \sim c_2(1 + 2\gamma f'(\alpha))\varepsilon_k^2.$$

ACCELERATION OF THE MODIFIED NEWTON METHOD

Formulae for accelerating modified Newton method are

$$\gamma_k = -\frac{1}{2f'(w_{k-1})} \qquad \textbf{Model (I)}, \tag{6}$$

$$\gamma_k = -\frac{1}{2f[x_k, x_{k-1}]} \qquad \textbf{Model (II)}, \tag{7}$$

$$\gamma_k = -\frac{1}{2(f'(w_{k-1}) + 2a_2(x_k - w_{k-1}))} \qquad \textbf{Model (III)}, \tag{8}$$

ACCELERATION OF THE MODIFIED NEWTON METHOD

(CONTINUATION)

where

$$P(t) = a_0 + a_1(t - x_k) + a_2(t - x_k)(t - w_{k-1})$$

is the interpolating polynomial that satisfies the interpolation conditions

$$P(x_k) = f(x_k), \quad P'(w_{k-1}) = f'(w_{k-1}), \quad P(x_{k-1}) = f(x_{k-1}).$$

NEW METHOD WITH MEMORY OF NEWTON TYPE

Substituting γ by γ_k calculated by one of methods I–III we obtain

$$\begin{cases} \gamma_0, \ \text{is given}, \quad \gamma_k \ \text{is calculated by one of (6)–(8) for } k \geq 1, \\ w_k = x_k + \gamma_k f(x_k), \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(w_k)} \ (k = 0, 1, \ldots). \end{cases} \tag{9}$$

CONVERGENCE THEOREMS FOR THE MODIFIED NEWTON METHO

Theorem 2. *If an initial approximation* $\boldsymbol{x_0}$ *is sufficiently close to a zero* $\boldsymbol{\alpha}$ *of* $\boldsymbol{f}$, *then the order of convergence of the modified Newton method with memory* (9), *where* $\boldsymbol{\gamma_k}$ *is calculated by* (6) *or* (7), *is* $\mathbf{1+\sqrt{2}\approx 2.414.}$

Theorem 3. *When an initial approximation* $\boldsymbol{x_0}$ *is sufficiently close to the sought simple zero* $\boldsymbol{\alpha}$ *of* $\boldsymbol{f}$, *the order of convergence of the modified Newton method with memory* (9)∧(8) *is not less than* $\mathbf{1+\sqrt{2}\approx 2.414,}$ *and with appropriate distribution of approximations it is not less than* $\mathbf{1+\sqrt{3}\approx 2.732.}$

COMPUTATIONAL ASPECTS

Computational efficiency of any iterative zero-finding method is closely connected to the features such as

- **the number of necessary numerical operations in computing zeros with the required accuracy,**
- **the convergence speed,**
- **processor time of a computer, etc.**

COMPUTATIONAL ASPECTS

Coefficient of efficiency is given by

$$E(IM) = r^{1/\theta}, \tag{10}$$

where

r **is the order of convergence of the iterative method** (IM)**, and**

θ **is the computational cost (number of function evaluations per iteration).**

COMPUTATIONAL ASPECTS

Using formula (10) we find

$$E(N) = E(T-S) = E(1) = E(5) = 2^{1/2} \approx 1.414,$$
$$E(4 \wedge 2) = 3^{1/2} \approx 1.732,$$
$$E(4 \wedge 3) = \left(\frac{1}{2}(3+\sqrt{17})\right)^{1/2} \approx 1.887,$$
$$E(T-S\ M) = (1+\sqrt{2})^{1/2} \approx 1.554,$$
$$E(T\ M) = E(9) = (1+\sqrt{3})^{1/2} \approx 1.653.$$

Computational efficiency of any optimal three-step scheme (IM) is $E(IM) = 8^{1/4} \approx 1.682,$

NUMERICAL EXAMPLES

We have tested the new methods (1), (4), (5) and (9), and the methods (N), $(T-S)$ and $(T\ M)$

We applied the programming package Mathematica with multiprecision arithmetic based on the GNU multiprecision package GMP developed by Granlund

T. Granlund, GNU MP; The GNU multiple precision arithmetic library, edition 5.0.1 (2010).

Test functions:

$$
\begin{aligned}
f_1(x) &= e^{-x^2+x+2} - \cos(x+1) + x^3 + 1,\ \alpha = -1,\ x_0 = -1.7,\\
&\quad \gamma_0 = -0.01,\ p_0 = -0.01,\\
f_2(x) &= (x-1)(x^6 + x^{-6} + 4)\sin x^2,\ \alpha = 1,\quad x_0 = 1.5,\\
&\quad \gamma_0 = -0.05,\ p_0 = 0,\\
f_3(x) &= \prod_{i=1}^{12}(x-i),\ \alpha = 8,\quad x_0 = 8.33,\ \gamma_0 = -0.1,\ p_0 = 0,\\
f_4(x) &= x + \sin x + \frac{1}{x} - 1 + 2i,\ \alpha \approx 0.2886 - i1.2422,\\
&\quad x_0 = -1 - 3i\ (i = \sqrt{-1}),\quad \gamma_0 = -0.05,\ p_0 = -0.05.
\end{aligned}
$$

NUMERICAL EXAMPLES

The tables contain the computational order of convergence r_c evaluated by the following formula

$$r_c = \frac{\log |f(x_k)/f(x_{k-1})|}{\log |f(x_{k-1})/f(x_{k-2})|}. \tag{11}$$

The errors $|x_k - \alpha|$ of approximations to the zeros are given in the form $A(-h)$, which denotes $A \times 10^{-h}$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c (11)
$(T-S)$	1.37(−1)	9.28(−4)	1.36(−7)	2.88(−15)	2.00
(1)	1.40(−1)	7.35(−4)	8.00(−8)	9.41(−16)	2.00
(4)∧(2)	1.37(−1)	5.81(−4)	4.75(−12)	2.87(−36)	3.00
(4)∧(3)	1.37(−1)	1.51(−4)	8.34(−15)	2.23(−51)	3.57
(N)	1.49(−1)	8.40(−4)	1.18(−7)	2.33(−15)	2.00
$(T\ M)$	1.49(−1)	1.98(−3)	8.96(−9)	3.48(−23)	2.70
(5)	1.24(−1)	9.16(−4)	1.24(−7)	2.24(−15)	2.00
(9)∧(6)	1.24(−1)	5.25(−4)	8.73(−10)	1.09(−23)	2.41
(9)∧(7)	1.24(−1)	3.67(−4)	3.26(−10)	1.09(−24)	2.38
(9)∧(8)	1.24(−1)	1.33(−5)	4.47(−13)	4.21(−35)	2.95

Table 1: One-point method without/with memory, test function f_1

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c (11)
$(T-S)$	1.04(−1)	1.19(−2)	1.42(−4)	1.94(−8)	2.00
(1)	1.00(−1)	1.06(−2)	1.03(−4)	9.35(−9)	2.00
(4)∧(2)	1.04(−1)	1.26(−3)	1.04(−8)	1.97(−24)	3.09
(4)∧(3)	1.04(−1)	2.65(−4)	1.55(−12)	4.31(−42)	3.59
(N)	9.98(−2)	1.57(−2)	3.37(−4)	1.46(−7)	2.01
$(T-M)$	9.98(−2)	2.90(−2)	8.56(−5)	1.16(−11)	2.73
(5)	8.44(−2)	2.99(−3)	5.73(−6)	2.09(−11)	2.00
(9)∧(6)	8.44(−2)	3.03(−3)	1.51(−6)	9.98(−15)	2.47
(9)∧(7)	8.44(−2)	3.10(−3)	1.05(−6)	5.71(−15)	2.38
(9)∧(8)	8.44(−2)	3.14(−3)	7.04(−7)	1.53(−16)	2.64

Table 2: One-point method without/with memory, test function f_2

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c (11)
$(T-S)$	div.				
(1) for all p_0	div.				
(4)∧(2)	3.30(−1)	3.30(−1)	1.29(−2)	7.64(−7)	3.00
(4)∧(3)	3.30(−1)	3.30(−1)	1.62(−2)	6.92(−8)	4.11
(N)	7.22(−2)	3.97(−3)	7.84(−6)	3.14(−11)	2.00
$(T\ M)$	div.				
(5)	div.				
(9)∧(6)	7.22(−2)	6.84(−4)	8.53(−9)	1.25(−20)	2.41
(9)∧(7)	7.22(−2)	1.13(−5)	2.93(−12)	2.52(−29)	2.59
(9)∧(8)	7.22(−2)	5.28(−4)	5.51(−10)	3.43(−24)	2.37

Table 3: One-point method without/with memory, test function f_3

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c (11)
$(T-S)$	$9.69(-1)$	$1.77(-1)$	$3.67(-3)$	$2.31(-6)$	1.89
(1)	$9.38(-1)$	$1.54(-1)$	$3.72(-3)$	$2.85(-6)$	1.92
(4)∧(2)	$9.38(-1)$	$8.37(-2)$	$3.32(-5)$	$1.06(-15)$	3.08
(4)∧(3)	$9.38(-1)$	$1.95(-2)$	$1.15(-10)$	$2.26(-33)$	2.76
(N)	$1.29(0)$	$4.95(-1)$	$1.95(-2)$	$7.51(-5)$	1.70
$(T\ M)$	$1.34(0)$	$1.48(-1)$	$3.05(-4)$	$1.88(-10)$	2.32
(5)	$7.29(-1)$	$6.71(-2)$	$5.61(-4)$	$4.30(-8)$	1.97
(9)∧(6)	$7.29(-1)$	$6.27(-2)$	$1.51(-4)$	$6.79(-11)$	2.42
(9)∧(7)	$7.29(-1)$	$5.78(-2)$	$9.29(-5)$	$2.00(-11)$	2.38
(9)∧(8)	$7.29(-1)$	$6.05(-2)$	$1.08(-4)$	$3.24(-12)$	2.74

Table 4: One-point method without/with memory, test function f_4

THANK YOU FOR YOUR ATTENTION!