FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 28, No 4 (2013), 429–441

# MODIFIED NEWTON'S METHOD WITH MEMORY *

**Jovana Džunić**

**Abstract.** The well-known Newton's iterative method for solving nonlinear equations is modified by introducing a free parameter. Applying an accelerating technique based on varying a free parameter calculated by Hermite-Birkhoff interpolating polynomial [1] in each iteration, the order of convergence of this method is considerably increased without additional computational cost. In this way, the reuse of old information provides a high computational efficiency. Numerical experiments confirm the theoretical results.

## 1. Preliminaries

Iterative methods with memory for solving nonlinear equations, that use information from the current and previous iteration, were considered for the first time by Traub in 1964 in his book [13] almost fifty years ago. Surprisingly enough, after Truab's research this class of methods was studied very seldom in the literature in spite of its capability to reach high computational efficiency. Recent results published in [3]–[4] and [2] showed considerably high computational efficiency of $n$-point methods with memory using new accelerating techniques based on varying free parameters calculated by interpolating polynomials in each iteration. In this paper we show that this accelerating approach can be successfully applied to a very familiar one-step method such as Newton's. High order of convergence is attained without additional function evaluations, making the mentioned methods very efficient. Numerical examples are given to demonstrate excellent convergence features of the presented methods with memory.

Let $\alpha$ be a simple real zero of a real function $f : D \subset \mathbb{R} \to \mathbb{R}$ and let $x_0$ be an initial approximation to $\alpha$. The well known Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = N(x_k) \tag{1.1}$$

Received November 11, 2013.
2000 *Mathematics Subject Classification*. Primary 65H05
*The authors were supported in part by Serbian Ministry of Science under grant 174022

and its derivative free variant proposed by Steffensen [12]

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)} \tag{1.2}$$

are the most well known one-step iterations of order two.

In his book [13] Traub considered the iterative function of order two

$$\Phi(x,\gamma) = x - \frac{\gamma f(x)^2}{f(x+\gamma f(x)) - f(x)} = x - \frac{f(x)}{f[x+\gamma f(x), x]}, \tag{1.3}$$

where $\gamma \neq 0$ is a real constant and $f[x,y] = \frac{f(x)-f(y)}{x-y}$ denotes a divided difference. Note that the choice $\gamma = 1$ reduces (1.3) to (1.2).

Introducing the abbreviations

$$u(x) = \frac{f(x)}{f'(x)} \quad \text{and} \quad C_2(x) = \frac{f''(x)}{2f'(x)},$$

Traub [13] derived the error relation of the method (1.3) in the form

$$\Phi(x,\gamma) - \alpha = (1 + \gamma f'(x))C_2(x)u(x)^2 + O\big(u(x)^3\big),$$

and showed that the Steffensen-like method (1.3) can somewhat be improved by the reuse of information from the previous iteration. Approximating $f'(x)$ by the secant

$$f'(x_k) \approx \widetilde{f'}(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f[x_k, x_{k-1}],$$

Traub [13] constructed the following method with memory with the order of convergence at least $1 + \sqrt{2} \approx 2.414$

$$\begin{cases} \gamma_0 \text{ is given}, \quad (k = 0, 1, \ldots) \\ \gamma_k = -\dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad \text{for } k \geq 1, \\ x_{k+1} = x_k - \dfrac{\gamma_k f(x_k)^2}{f(x_k + \gamma_k f(x_k)) - f(x_k)}, \end{cases}$$

In the same book [13], Traub considered some other techniques of acceleration with the reuse of information from the previous iteration. Among others he studied the accelerated Newton's method of order $1 + \sqrt{3} \approx 2.732$. In fact, Traub considered the Halley's method

$$x_{k+1} = x_k - \frac{u(x_k)}{1 - C_2(x_k)u(x_k)}$$

and eliminated the second derivative to construct the iterative scheme

$$\begin{cases} p_0 \text{ is given}, \quad (k = 0, 1, \ldots), \\ p_k = -\dfrac{H_3''(x_k)}{2f'(x_k)} \quad \text{for } k \geq 1, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k) + p_k f(x_k)}, \end{cases} \tag{1.4}$$

where $H_3(t) = H_3(t; x_k, x_k, x_{k-1}, x_{k-1})$ is Hermite's interpolating polynomial of third degree that satisfies interpolating conditions $H_3(x_i) = f(x_i)$, $H_3'(x_i) = f'(x_i)$, $i = k, k-1$. Thus

$$H_3''(x_k) = \frac{2(2f'(x_k) + f'(x_{k-1}) - 3f[x_k, x_{k-1}])}{x_k - x_{k-1}}.$$

Similar approaches for accelerating derivative free multipoint methods by varying parameters were applied in [6], [5], [4] and [2] in a considerably more efficient way. Following Traub's classification [13, pp. 8-9], methods that use information from the current and previous iteration are called *methods with memory*. Presenting a new one-step method with memory is the aim of this paper.

In our convergence analysis we employ the $O$- and $o$-notation: If $\{g_k\}$ and $\{h_k\}$ are null sequences and $g_k/h_k \to C$, where $C$ is a nonzero constant, we write $g_k = O(h_k)$ or $g_k \sim Ch_k$. If $g_k/h_k \to 0$, we write $g_k = o(h_k)$.

Let $\{x_k\}$ be a sequence of approximations generated by an iterative method ($IM$). We introduce error $\varepsilon_k = x_k - \alpha$. If this sequence converges to the zero $\alpha$ of $f$ with the order $r$, we will write

$$\varepsilon_{k+1} \sim D_{k,r}\varepsilon_k^r, \tag{1.5}$$

where $D_{k,r}$ tends to the asymptotic error constant $D_r$ of the iterative method ($IM$) when $k \to \infty$. Formally, we use the order of a sequence of approximations as the subscribe index to distinguish asymptotic error constants.

## 2. Modified Newton's method

In 1964's book [13], Traub derived a number of non-optimal cubically convergent two-point methods. He used an interpolating function to obtain the following implicit relation in $\alpha$,

$$\alpha \approx x - \frac{f(x)}{f'\Big(x + \frac{1}{2}(\alpha - x)\Big)}. \tag{2.1}$$

Substituting $\alpha$ on the right-hand side of (2.1) by Newton's approximation $x - u(x)$, the following iterative function of order three is obtained

$$\Phi(x) = x - \frac{f(x)}{f'(x - \frac{1}{2}u(x))},$$

requiring three function evaluations.

Assuming that $\alpha - x = O(f(x))$ and substituting $\alpha - x = 2\gamma f(x)$ in (2.1), we obtain the following modification of Newton's method (1.1)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(w_k)}, \tag{2.2}$$

where $w_k = x_k + \gamma f(x_k)$. The new method (2.2) uses one function and one derivative evaluation. Standard analysis based on Taylor's expansions

$$f(x_k) = f'(\alpha)\Big(\varepsilon_k + c_2\varepsilon_k^2 + O(\varepsilon_k^3)\Big), \tag{2.3}$$

$$\varepsilon_{k,w} = \varepsilon_k(1 + \gamma f'(\alpha)) + O(\varepsilon_k^2), \tag{2.4}$$

$$f'(w_k) = f'(\alpha)\Big(1 + 2c_2\varepsilon_{k,w} + O(\varepsilon_{k,w}^2)\Big), \tag{2.5}$$

$$\frac{f(x_k)}{f'(w_k)} = \varepsilon_k\Big(1 + c_2\varepsilon_k + O(\varepsilon_k^2)\Big)\Big(1 - 2c_2\varepsilon_{k,w} + O(\varepsilon_{k,w}^2)\Big), \tag{2.6}$$

leads to the error relation for the new method (2.2)

$$\varepsilon_{k+1} = c_2\varepsilon_k(2\varepsilon_{k,w} - \varepsilon_k) + O(\varepsilon_k^3) = c_2(1 + 2\gamma f'(\alpha))\varepsilon_k^2 + O(\varepsilon_k^3). \tag{2.7}$$

Thus, the method (2.2) is of order two as the original Newton's method (1.1). However, the error relation (2.7) of the new method (2.2) gives a way for an improvement in convergence speed without additional function evaluations. Such acceleration is discussed in Section 3..

Another interpretation of method (2.2) can be given by the following observation. According to Lagrange's Theorem, there exists a point $z$ (see Figure 2.1), between the zero $\alpha$ and its approximation $x$, with a tangent line $t$ parallel to the secant $s : \overline{(\alpha, 0)(x, f(x))}$.

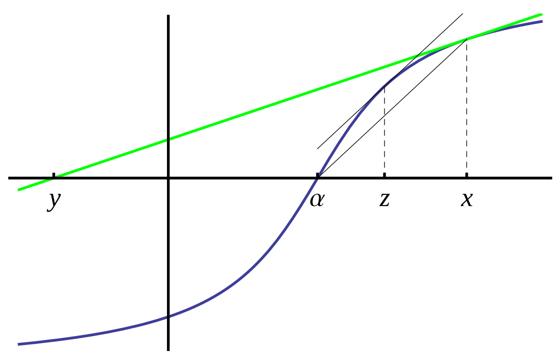


Fig. 2.1: Geometric interpretation of method (2.2)

Thus, from Lagrange's relation we obtain

$$\frac{f(x) - f(\alpha)}{x - \alpha} = f'(z),$$

which results in

$$\alpha = x - \frac{f(x)}{f'(z)}.$$

Our best guess is to take $z = x+\gamma f(x)$, for some choice of $\gamma$, which yields the method (2.2), again.

As shown on Figure 2.1, Newton's approximation $y = N(x)$ cannot provide a quality approximation for the specific considered case. The sought zero is overshoot by the tangent line at $(x, f(x))$ (in green). With an adequate choice of $\gamma$ the problem is overcome by the method (2.2).

## 3. Acceleration of the modified Newton's method

The modified Newton method (2.2) can be accelerated with the use of information from the previous iteration. Minimization of the error relation (2.7) is obtained by recalculating the free parameter $\gamma = \gamma_k \approx -1/(2f'(\alpha))$ in each iterative step. We propose the following formulae for approximating $f'(\alpha)$ based on the available data,

$$f'(\alpha) \approx f'(w_{k-1}), \tag{3.1}$$

$$f'(\alpha) \approx f[x_k, x_{k-1}] = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}, \tag{3.2}$$

$$f'(\alpha) \approx P'(x_k), \tag{3.3}$$

where

$$P(t) = a_0 + a_1(t - x_k) + a_2(t - x_k)(t - w_{k-1})$$

is the interpolating polynomial of Hermite-Birkhoff type [1] that satisfies the following interpolation conditions

$$P(x_k) = f(x_k), \quad P'(w_{k-1}) = f'(w_{k-1}), \quad P(x_{k-1}) = f(x_{k-1}). \tag{3.4}$$

From the conditions (3.4) we obtain coefficients $a_0$, $a_1$ and $a_2$ for the polynomial $P$

$$\begin{gathered} a_0 = f(x_k), \quad a_1 = f'(w_{k-1}) + a_2(x_k - w_{k-1}), \\ a_2 = \frac{f[x_k, x_{k-1}] - f'(w_{k-1})}{x_k + x_{k-1} - 2w_{k-1}}. \end{gathered} \tag{3.5}$$

Based on the relations (3.1)–(3.3), formulae for the accelerating modified Newton method are of the form

$$\text{Model I} \qquad \gamma_k = -\frac{1}{2f'(w_{k-1})}, \tag{3.6}$$

$$\text{Model II} \qquad \gamma_k = -\frac{1}{2f[x_k, x_{k-1}]}, \tag{3.7}$$

$$\text{Model III} \qquad \gamma_k = \frac{-1/2}{f'(w_{k-1}) + 2a_2(x_k - w_{k-1})} \tag{3.8}$$

with the coefficient $a_2$ given in (3.5).

Combining (2.2) with (3.6)–(3.8), we construct the modified Newton method with memory

$$(3.9)\qquad \begin{cases} \gamma_0 \text{ is given,} \\ \gamma_k \text{ is calculated by one of (3.6)–(3.8)},\ k \geq 1, \\ w_k = x_k + \gamma_k f(x_k), \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(w_k)}\ (k = 0, 1, \ldots). \end{cases}$$

New error relations obtained from (2.4) and (2.7) read

$$(3.10)\qquad \varepsilon_{k,w} \sim (1 + \gamma_k f'(\alpha))\varepsilon_k,$$

$$(3.11)\qquad \varepsilon_{k+1} \sim c_2(1 + 2\gamma_k f'(\alpha))\varepsilon_k^2.$$

**Theorem 3.1.** *If an initial approximation $x_0$ is sufficiently close to a zero $\alpha$ of $f$, then the order of convergence of the modified Newton method with memory* (3.9)*, where $\gamma_k$ is calculated by* (3.6) *or* (3.7)*, is* $1 + \sqrt{2} \approx 2.414$.

*Proof.* Note that acceleration technique of minimizing the term $1+2\gamma f'(\alpha)$ does not influence the approximation $w_k$ since its error is

$$(3.12)\qquad \varepsilon_{k,w} \sim (1 + \gamma_k f'(\alpha))\varepsilon_k = \left(\tfrac{1}{2} + O(\varepsilon_{k-1}^q)\right)\varepsilon_k = O(\varepsilon_k),$$

therefore, still of order $\varepsilon_k$.

According to (2.5), (3.6) and (3.12) we have

$$\begin{aligned} 1 + 2\gamma_k f'(\alpha) &= 1 - \frac{f'(\alpha)}{f'(w_{k-1})} \\ &= 1 - \frac{1}{1 + 2c_2\varepsilon_{k-1,w} + O(\varepsilon_{k-1,w}^2)} \\ (3.13)\qquad &\sim 2c_2\varepsilon_{k-1,w} \sim c_2\varepsilon_{k-1}. \end{aligned}$$

If (3.7) is employed to recalculate $\gamma_k$, the following estimates are obtained similar to (3.13)

$$\begin{aligned} f[x_k, x_{k-1}] &\sim f'(\alpha)(1 + c_2\varepsilon_{k-1}), \\ (3.14)\qquad 1 + 2\gamma_k f'(\alpha) &= 1 - \frac{f'(\alpha)}{f[x_k, x_{k-1}]} \sim 1 - \frac{1}{1 + c_2\varepsilon_{k-1}} \sim c_2\varepsilon_{k-1}. \end{aligned}$$

Therefore, both accelerating methods given by formulae (3.6) and (3.7) give approximations of the same quality.

After substituting (3.13) or (3.14) and (1.5) into (3.11), and having in mind (3.12), we estimate

$$(3.15)\qquad \varepsilon_{k+1} \sim 2c_2^2\varepsilon_{k-1,w}\varepsilon_k^2 = O(\varepsilon_{k-1}^{2r+1}).$$

Comparing exponents of the term $\varepsilon_{k-1}$ in the relations (3.15) and

$$\varepsilon_{k+1} \sim D_{k,r} D_{k-1,r}^{r} \varepsilon_{k-1}^{r^2} = O\Big(\varepsilon_{k-1}^{r^2}\Big) \tag{3.16}$$

yields the equation $r^2 - 2r - 1 = 0$. Its unique positive solution gives the order of convergence $r = 1 + \sqrt{2}$ of the modified Newton methods with memory (3.9)∧(3.6) and (3.9)∧(3.7). □

To find the convergence rate of the method (3.9)∧(3.8) we need an error estimate of the approximating polynomial $P$.

**Lemma 3.1.** *If $\gamma_k$ is calculated by (3.8), than the estimate $1+2\gamma_k f'(\alpha) = O(\varepsilon_{k-1})$ holds. In particular, when the node $w_{k-1}$ is not between nodes $x_k$ and $x_{k-1}$, then at least $1+2\gamma_k f'(\alpha) = O(\varepsilon_{k-1}^2)$.*

*Proof.* From the interpolation conditions (3.4) and Taylor's expansion it follows

$$\begin{aligned} P'(x_k) &= P'(w_{k-1}) + P''(w_{k-1})(\varepsilon_{k-1,w} - \varepsilon_k) + O\Big((\varepsilon_{k-1,w} - \varepsilon_k)^2\Big) \\ &= f'(w_{k-1}) + O(\varepsilon_{k-1,w}) = f'(\alpha) + O(\varepsilon_{k-1,w}). \end{aligned}$$

Hence, we have

$$1 + 2\gamma_k f'(\alpha) = 1 - \frac{f'(\alpha)}{P'(x_k)} = O(\varepsilon_{k-1,w}) = O(\varepsilon_{k-1}). \tag{3.17}$$

Let us now explore the case when $w_{k-1}$ is not between $x_k$ and $x_{k-1}$. We introduce a function $F$ as the difference

$$F(t) = f(t) - P(t).$$

According to Roll's Theorem, there exists a point

$$\zeta_1 = x_{k-1} + \theta_1(x_k - x_{k-1}),\ \ \theta_1 \in (0,1),$$

between $x_k$ and $x_{k-1}$ (different from $w_{k-1}$) such that $F'(\zeta_1) = 0$. Once again due to Roll's Theorem, there exists a point

$$\zeta_2 = w_{k-1} + \theta_2(\zeta_1 - w_{k-1}),\ \ \theta_2 \in (0,1)$$

such that $F''(\zeta_2) = 0$. Therefore, at least $\varepsilon_{\zeta_j} = \zeta_j - \alpha = O(\varepsilon_{k-1})$ holds for $j = 1, 2$.

Estimating the difference $P'(x_k) - f'(x_k)$ by Taylor's series, we obtain

$$\begin{aligned} P'(x_k) - f'(x_k) &= F'(x_k) = F''(\zeta_1)(\zeta_1 - x_k) + O\Big((\zeta_1 - x_k)^2\Big) \\ &= \Big(F'''(\zeta_2)(\zeta_2 - \zeta_1) + O\Big((\zeta_2 - \zeta_1)^2\Big)\Big)(\zeta_1 - x_k) + O\Big((\zeta_1 - x_k)^2\Big) \\ &= f'''(\zeta_2)(\varepsilon_{\zeta_2} - \varepsilon_{\zeta_1}) + O\Big((\varepsilon_{\zeta_2} - \varepsilon_{\zeta_1})^2\Big)\Big)(\varepsilon_{\zeta_1} - \varepsilon_k) + O\Big((\varepsilon_{\zeta_1} - \varepsilon_k)^2\Big) \\ &= O(\varepsilon_{k-1}^2)\ \ \text{(at least)}, \end{aligned}$$

that is, according to (2.3) and (1.5),

$$P'(x_k) = f'(\alpha) + O(\varepsilon_{k-1}^2) \text{ (at least).} \tag{3.18}$$

From (3.18) than it easily follows

$$1 + 2\gamma_k f'(\alpha) = 1 - \frac{f'(\alpha)}{P'(x_k)} = O\left(\varepsilon_{k-1}^2\right). \tag{3.19}$$

□

Conclusions from the previous Lemma coincide with the spline form of an error relation of the Hermit-Birkhoff interpolation [1].

Now we can state the following Theorem.

**Theorem 3.2.** *When an initial approximation $x_0$ is sufficiently close to the sought simple zero $\alpha$ of $f$, the order of convergence of the modified Newton method with memory* (3.9)∧(3.8) *is not less than* $1 + \sqrt{2} \approx 2.414$*, and with appropriate distribution of approximations it is not less than* $1 + \sqrt{3} \approx 2.732$.

*Proof.* Let $I = (\min\{x_{k-1}, x_k\}, \max\{x_{k-1}, x_k\})$. Similar to the proof of Theorem 3.1, using relations from Lemma 3.1, (3.17) and (3.19) in the error relation (3.11), we obtain quadratic equations that define orders of convergence of the modified method with memory (3.9)∧(3.8)

$$r^2 - 2r - 1 = 0, \qquad r = 1 + \sqrt{2}, \qquad w_{k-1} \in I,$$

and

$$r^2 - 2r - 2 = 0, \qquad r = 1 + \sqrt{3}, \qquad w_{k-1} \notin I.$$

□

Although the error relation (3.10) holds and contains the term $1 + \gamma_k f'(\alpha)$, from the error relation (3.11) we conclude that the improvement of approximation $w_k$ (as it was done in [2] and [6]), does not result in improving $x_{k+1}$ which is the ultimate goal.

## 4. Numerical results

The computational efficiency of an iterative method (IM) of the order $r$, requiring $\theta$ new function evaluations per iteration, is usually calculated by Ostrowski-Traub's formula

$$E(IM) = r^{1/\theta}$$

(see [10, p. 20], [13, Appendix C]). Using this formula we find

$$
\begin{aligned}
E(1.1) &= E(1.3) = 2^{1/2} \approx 1.414,\\
E(3.9) &= (1+\sqrt{2})^{1/2} \approx 1.554,\\
E(1.4) &= E(3.9) = (1+\sqrt{3})^{1/2} \approx 1.653.
\end{aligned}
$$

where the last entry is obtained only with a special distribution of approximations. Recall that computational efficiency of any optimal three-step scheme ($IM$) is $E(IM) = 8^{1/4} \approx 1.682$, see, for example, [11]. The obtained efficiency indices of the proposed methods are very high.

**Remark 4.1.** We note that according to Herzberger's matrix method [9], a one-step interpolatory iteration function with memory, which uses all the available information from the previous and actual iteration, obtains the order of convergence at most $\frac{1}{2}(3+\sqrt{17})$ for iterations based on $f$ evaluation, and $1+\sqrt{3}$ for iterations based on $f$ and $f'$ evaluation.

We have tested the new methods (2.2) and (3.9), along with the methods (1.1), (1.3), and (1.4) using the computational software package *Mathematica* with multiple-precision arithmetic. We selected the following four test functions:

$$
\begin{aligned}
f_1(x) &= e^{-x^2+x+2} - \cos(x+1) + x^3 + 1,\ \alpha = -1,\ x_0 = -1.7,\\
&\quad \gamma_0 = -0.01,\ p_0 = -0.01,
\end{aligned}
$$

$$
f_2(x) = (x-1)(x^6 + x^{-6} + 4)\sin x^2,\ \alpha = 1,\ \ x_0 = 1.5,\ \gamma_0 = -0.05,\ p_0 = 0,
$$

$$
f_3(x) = \prod_{i=1}^{12}(x-i),\ \alpha = 8,\ \ x_0 = 8.33,\ \gamma_0 = 0^{1},\ p_0 = 0,
$$

$$
\begin{aligned}
f_4(x) &= x + \sin x + \frac{1}{x} - 1 + 2i, \alpha \approx 0.2886 - i1.2422,\ x_0 = -1-3i\ (i=\sqrt{-1}),\\
&\quad \gamma_0 = -0.05,\ p_0 = -0.05.
\end{aligned}
$$

The plot of the function $f_2$ is shown in Figure 4.1. The function shows nontrivial behavior since the graph is of Π-form and with a singularity and another zero close to the sought zero.

Test function $f_3$ is a polynomial of Wilkinson's type with real zeros $1, 2, \ldots, 12$. A well-known fact that these polynomials are ill-conditioned motivated us to choose this test function. Wilkinson's polynomials coefficients are of order of magnitude of factorials of its zeroes, see Figure 4.2. This is the reason why small perturbations of polynomial coefficients lead to drastic variations of polynomial zeros.

Complex test function $f_4$ is used to show that the proposed methods are, as expected, applicable in complex domain, even though entire convergence analysis

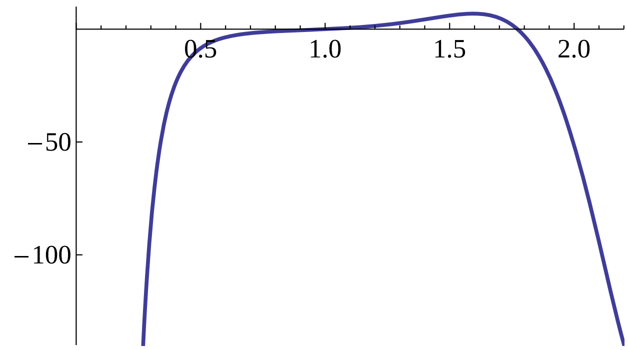


FIG. 4.1: Graph of the function $f_2(x)$

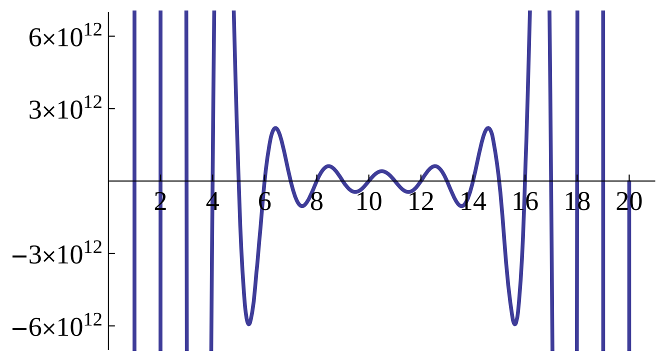


FIG. 4.2: Graph of the function $f_3(x)$

was undertaken under an assumption of the real domain. As noted by Geum and Kim in [8], $f_4$-kind of functions arise from the real life problems related to steady-state heat flow, electrostatic potential and fluid flow.

We note that any multipoint method that uses Newton's method (1.1) or Steffensen's method (1.2) as predictor steps, is unusual to show global convergence qualities. However, slight improvement in basins of attraction can be achieved with predictor methods such as (3.3) and methods from [6], by an adequate choice of initial values for the parameters.

Very accurate initial approximations to the sought zero can be found using recently developed efficient Yun's method [15] based on numerical integration. In this manner, information of the function $f$, used for determining the initial value $x_0$ by the mentioned numerical integration, can be successively used for determining a good starting value for $\gamma_0$.

The errors $|x_k - \alpha|$ of approximations to the zeros are given in Tables 1–4, where $A(-h)$ denotes $A \times 10^{-h}$. It is evident that approximations to the roots given in Tables 4.1-4.4 possess great accuracy. Results of the fourth iteration are given only

Table 4.1: One-point method without/with memory, test function $f_1$

| Methods | $\|x_1-\alpha\|$ | $\|x_2-\alpha\|$ | $\|x_3-\alpha\|$ | $\|x_4-\alpha\|$ | $r_c$ (4.1) |
|---|---|---|---|---|---|
| (1.3) | 1.37(−1) | 9.28(−4) | 1.36(−7) | 2.88(−15) | 2.00 |
| (1.1) | 1.49(−1) | 8.40(−4) | 1.18(−7) | 2.33(−15) | 2.00 |
| (1.4) | 1.49(−1) | 1.98(−3) | 8.96(−9) | 3.48(−23) | 2.70 |
| (2.2) | 1.24(−1) | 9.16(−4) | 1.24(−7) | 2.24(−15) | 2.00 |
| (3.9)∧(3.6) | 1.24(−1) | 5.25(−4) | 8.73(−10) | 1.09(−23) | 2.41 |
| (3.9)∧(3.7) | 1.24(−1) | 3.67(−4) | 3.26(−10) | 1.09(−24) | 2.38 |
| (3.9)∧(3.8) | 1.24(−1) | 1.33(−5) | 4.47(−13) | 4.21(−35) | 2.95 |

Table 4.2: One-point method without/with memory, test function $f_2$

| Methods | $\|x_1-\alpha\|$ | $\|x_2-\alpha\|$ | $\|x_3-\alpha\|$ | $\|x_4-\alpha\|$ | $r_c$ (4.1) |
|---|---|---|---|---|---|
| (1.3) | 1.04(−1) | 1.19(−2) | 1.42(−4) | 1.94(−8) | 2.00 |
| (1.1) | 9.98(−2) | 1.57(−2) | 3.37(−4) | 1.46(−7) | 2.01 |
| (1.4) | 9.98(−2) | 2.90(−2) | 8.56(−5) | 1.16(−11) | 2.73 |
| (2.2) | 8.44(−2) | 2.99(−3) | 5.73(−6) | 2.09(−11) | 2.00 |
| (3.9)∧(3.6) | 8.44(−2) | 3.03(−3) | 1.51(−6) | 9.98(−15) | 2.47 |
| (3.9)∧(3.7) | 8.44(−2) | 3.10(−3) | 1.05(−6) | 5.71(−15) | 2.38 |
| (3.9)∧(3.8) | 8.44(−2) | 3.14(−3) | 7.04(−7) | 1.53(−16) | 2.64 |

for demonstration of convergence speed of the tested methods and in most cases they are not required for practical problems at present. These tables include the values of the computational order of convergence $r_c$ calculated by the formula

$$r_c = \frac{\log|f(x_k)/f(x_{k-1})|}{\log|f(x_{k-1})/f(x_{k-2})|}, \tag{4.1}$$

taking into consideration the last three approximations in the iterative process.

In Table 4.3 *div.* means that the method in the row is not convergent to the zero $\alpha = 8$. However, for the given initial value these methods neither converge to any other zero of the Wilkinson's polynomial.

Table 4.3: One-point method without/with memory, test function $f_3$

| Methods | $\lvert x_1-\alpha\rvert$ | $\lvert x_2-\alpha\rvert$ | $\lvert x_3-\alpha\rvert$ | $\lvert x_4-\alpha\rvert$ | $r_c$ (4.1) |
|---|---|---|---|---|---|
| (1.3) | div. | | | | |
| (1.1) | 7.22(−2) | 3.97(−3) | 7.84(−6) | 3.14(−11) | 2.00 |
| (1.4) | div. | | | | |
| (2.2) | div. | | | | |
| (3.9)∧(3.6) | 7.22(−2) | 6.84(−4) | 8.53(−9) | 1.25(−20) | 2.41 |
| (3.9)∧(3.7) | 7.22(−2) | 1.13(−5) | 2.93(−12) | 2.52(−29) | 2.59 |
| (3.9)∧(3.8) | 7.22(−2) | 5.28(−4) | 5.51(−10) | 3.43(−24) | 2.37 |

Table 4.4: One-point method without/with memory, test function $f_4$

| Methods | $\lvert x_1-\alpha\rvert$ | $\lvert x_2-\alpha\rvert$ | $\lvert x_3-\alpha\rvert$ | $\lvert x_4-\alpha\rvert$ | $r_c$ (4.1) |
|---|---|---|---|---|---|
| (1.3) | 9.69(−1) | 1.77(−1) | 3.67(−3) | 2.31(−6) | 1.89 |
| (1.1) | 1.29(0) | 4.95(−1) | 1.95(−2) | 7.51(−5) | 1.70 |
| (1.4) | 1.34(0) | 1.48(−1) | 3.05(−4) | 1.88(−10) | 2.32 |
| (2.2) | 7.29(−1) | 6.71(−2) | 5.61(−4) | 4.30(−8) | 1.97 |
| (3.9)∧(3.6) | 7.29(−1) | 6.27(−2) | 1.51(−4) | 6.79(−11) | 2.42 |
| (3.9)∧(3.7) | 7.29(−1) | 5.78(−2) | 9.29(−5) | 2.00(−11) | 2.38 |
| (3.9)∧(3.8) | 7.29(−1) | 6.05(−2) | 1.08(−4) | 3.24(−12) | 2.74 |

## REFERENCES

1. G. D. Birkhoff: *General mean-value and remainder theorems with applications to mechanical differentiation and integration*. Trans. Amer. Math. Soc. **1** (1906), 107–136.
2. J. Džunić, M. S. Petković and L. D. Petković: *A family of two-point methods with memory for solving nonlinear equations*. Appl. Anal. Discrete Math. **5** (2011), 298–317.
3. J. Džunić: *Multistep Methods for Solving Nonlinear Equations*. Ph. D. Thesis, Faculty of Science and Mathematics, University of Niš, Niš, 2012.
4. J. Džunić, M. S. Petković and L. D. Petković: *Three-point methods with and without memory for solving nonlinear equations*. Appl. Math. Comput., **218** (2012), 4917–4927.
5. J. Džunić and M. S. Petković: *On generalized multipoint root-solvers with memory*. J. Comput. Appl. Math. **236** (2012), 2909–2920.
6. J. Džunić: *On efficient two-parameter methods for solving nonlinear equations*. Numer. Algor.**63** (2013), 549–569.
7. J. Džunić and M. S. Petković: *On generalized biparametric multipoint root finding methods with memory*. J. Comput. Appl. Math., **255** (2014), 362–375.
8. Y. H. Geum and Y. I. Kim: *A multi-parameter family of three-step eight-order iterative methods locating a simple root*. Appl. Math. Comput. **215** (2010), 3375–3382.
9. J. Herzberger: *Über Matrixdarstellungen für Iterationverfahren bei nichtlinearen Gleichungen*. Computing **12** (1974), 215–222.

10. A. Ostrowski: *Solution of Equations and Systems of Equations*. Academic Press, New York, 1966.

11. M. S. Petković, B. Neta, L. D. Petković and J. Džunić: *Multipoint Methods for Solving Nonlinear Equations*. Elsevier, Amsterdam-Boston-Heidelberg, 2013.

12. I. F. Steffensen: *Remarks on iteration*. Skand. Aktuarietidskr. **16** (1933), 64–72.

13. J. F. Traub: *Iterative Methods for the Solution of Equations*. Prentice-Hall, New Jersey, 1964.

14. J. H. Wilkinson: *Rounding Errors in Algebraic Processes*. Prentice Hall, New Jersey, 1963.

15. B.I. Yun: *A non-iterative method for solving nonlinear equations*. Appl. Math. Comp. **198** (2008), 691–699.

Jovana Džunić
Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
jovana.dzunic@elfak.ni.ac.rs