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Research Article

A Family of Three-Point Methods of Ostrowski's Type for Solving Nonlinear Equations

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A class of three-point methods for solving nonlinear equations of eighth order is constructed. These methods are developed by combining two-point Ostrowski's fourth-order methods and a modified Newton's method in the third step, obtained by a suitable approximation of the first derivative using the product of three weight functions. The proposed three-step methods have order eight costing only four function evaluations, which supports the Kung-Traub conjecture on the optimal order of convergence. Two numerical examples for various weight functions are given to demonstrate very fast convergence and high computational efficiency of the proposed multipoint methods.

1. Introduction

Multipoint methods for solving nonlinear equations $f(x) = 0$, where $f : D \subset \mathbf{R} \to \mathbf{R}$, possess an important advantage since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. More details may be found in the book [1] and many papers published in the first decade of the 21st century. In this paper we present a new family of three-point methods which employs Ostrowski's method in the first two steps and suitably chosen weight functions in the third step. The order of this family is eight requiring four function evaluations.

We start with a three-step scheme (omitting iteration index for simplicity)

$$\begin{aligned} y &= x - \frac{f(x)}{f'(x)}, \\ z &= y - \frac{f(y)}{f'(x)} \cdot \frac{f(x)}{f(x) - 2f(y)}, \\ \hat{x} &= z - \frac{f(z)}{f'(z)}, \end{aligned} \tag{1.1}$$

where x is a current approximation and $\hat{x}$ is a new approximation to a simple real zero α of f. Note that the first two steps form Ostrowski's two-point method [2] of order four.

The iterative method (1.1) has order eight but it requires five function evaluations, which is expensive from the computational point of view. To decrease this cost from 5 to 4 function evaluations, we want to approximate $f'(z)$ in the third step of (1.1) using available data $f(x)$, $f'(x)$, $f(y)$, $f(z)$. We are seeking this approximation in the form

$$f'(z) \approx f'(x)\phi(t)\psi(s)\omega(v), \tag{1.2}$$

where ϕ, ψ, and ω are sufficiently differentiable real-valued functions with the arguments

$$t = \frac{f(y)}{f(x)}, \qquad s = \frac{f(z)}{f(y)}, \qquad v = \frac{f(z)}{f(x)}. \tag{1.3}$$

Now the iterative scheme (1.1) becomes

$$\begin{aligned} y &= x - \frac{f(x)}{f'(x)}, \\ z &= y - \frac{f(y)}{f'(x)} \cdot \frac{f(x)}{f(x) - 2f(y)}, \\ \hat{x} &= z - \frac{f(z)}{f'(x)\phi(t)\psi(s)\omega(v)}. \end{aligned} \tag{1.4}$$

Functions ϕ, ψ, and ω should be determined in such a way that the iterative method (1.4) attains the order eight. Such procedure will be presented in Section 2.

2. Construction and Convergence of New Three-Point Root Solvers

To find the weight functions ϕ, ψ, and ω in (1.4) providing order eight, we will use the method of undetermined coefficients and Taylor's series about 0 since $t \to 0$, $s \to 0$, and $v \to 0$ when $x \to 0$. We have

$$\begin{aligned} \phi(t) &= \phi(0) + \phi'(0)t + \frac{\phi''(0)}{2!}t^2 + \frac{\phi'''(0)}{3!}t^3 + \cdots, \\ \psi(v) &= \psi(0) + \psi'(0)s + \cdots, \\ \omega(w) &= \omega(0) + \omega'(0)v + \cdots. \end{aligned} \tag{2.1}$$

The simplest method for finding the coefficients of the above Taylor expansions is the use of symbolic computation by a computer algebra system and an interactive procedure (comments C1–C4 in Algorithm 1), as already carried out for some of the previously developed methods, see, for example, [3]. The corresponding program can always display any desired formula or expression, although these expressions are cumbersome and only of academic interest.

```
fx=f1a*(e+c2 e^2+c3 e^3+c4 e^4); f1x=D[fx,e];
ey=e-Series[fx/f1x,{e,0,8}];
fy=f1a*(ey+c2 ey^2+c3 ey^3+c4 ey^4);
t=Series[fy/fx,{e,0,8}];
ez=ey-Series[1/(1-2t)*fy/f1x,{e,0,8}];
fz=f1a*(ez+c2 ez^2+c3 ez^3);
s=Series[fz/fy,{e,0,8}];
v=Series[fz/fx,{e,0,8}];
gt=t0+t10*t+t20*t^2/2+t30*t^3/6;
gs=s0+s10*s+s20*s^2/2;
gv=v0+v10*v+v20*v^2/2;
f1z=f1x*gt*gs*gv;
e1=ez-Series[fz/f1z,{e,0,8}]//Simplify
C1: "Out[a4]= c2(c2^2 - c3)(-1+t0 s0 v0)/(t0 s0 v0)"
t0=1; s0=1; v0=1; a5=Coefficient[e1,e^5] //Simplify
C2: "Out[a5] = c2^2(c2^2 - c3)(2 + t10)"
t10=-2; a6=Coefficient[e1,e^6]//Simplify
C3: "Out[a6] = 1/2 c2(c2^2 - c3)(-2c3 (1+s10) + c2^2(4 + t20 + 2v10))"
s10=-1; t20=-2; a7=Coefficient[e1,e^7]//Simplify
C4: "Out[a7] = 1/6 c2^2(c2^2 - c3)(-6c3(2+ v10) + c2^2(t30+6(2+v10)))"
v10=-2; t30=0; a8=Coefficient[e1,e^8]//Simplify
C5: "Out[a8] = 1/2 c2(c2^2 - c3)(2c2c4 + c3^2v20 + c2^4(4 + s20) - 2c2^2c3(4 + s20))e^8 + O[e^9]"
```

Algorithm 1: Program (written in *Mathematica*).

We introduce the following abbreviations:

ck= $f^{(k)}(\alpha)/(k!f'(\alpha))$, fx= $f(x)$,
f1x= $f'(x)$, f1a= $f'(\alpha)$, fy= $f(y)$, fz= $f(z)$,
e= $x-\alpha$, ey= $y-\alpha$, ez= $z-\alpha$, e1= $\hat{x}-\alpha$,
t0= $\psi(0)$, t10= $\psi'(0)$, t20= $p''(0)$, t30= $\psi'''(0)$,
s0= $\psi(0)$, s10= $\psi'(0)$,
v0= $\omega(0)$, v10= $\omega'(0)$.

Comment 1. C1: from the expression of the error $\hat{\varepsilon} = \hat{x} - \alpha$ we observe that $\hat{\varepsilon}$ is of the form

$$\hat{\varepsilon} = a_4\varepsilon^4 + a_5\varepsilon^5 + a_6\varepsilon^6 + a_7\varepsilon^7 + a_8\varepsilon^8 + O\left(\varepsilon^9\right). \tag{2.2}$$

The iterative three-point method (1.4) will have the order of convergence equal to eight if we determine the coefficients of the developments appearing in (2.1) in such way that a_4, a_5, a_6, a_7 (in (2.2)) all vanish. We find these coefficients equating shaded expressions in boxed formulas to 0. First, from Out[a4] we have

$$-1 + \phi(0)\psi(0)\omega(0) = 0. \tag{2.3}$$

Without the loss of generality, we can take $\phi(0) = \psi(0) = \omega(0) = 1$ with the benefit that the term $\phi(0)\psi(0)\omega(0)$ becomes 1 simplifying subsequent expressions.

In what follows, equating coefficient a_5, a_6, a_7 to 0, one obtains

C2: $\phi'(0) + 2 = 0 \Longrightarrow \phi'(0) = -2$,

C3: $\psi'(0) + 1 = 0 \quad \wedge \quad \phi''(0) + 2\psi'(0) + 4 = 0 \Longrightarrow \psi'(0) = -1, \phi''(0) = -2$,

C4: $\omega'(0) + 2 = 0 \wedge \phi'''(0) + 6(\omega'(0) + 2) = 0 \Longrightarrow \omega'(0) = -2, \phi'''(0) = 0$.

Comment 2. C5: substituting the quantities $\phi(0), \phi'(0), \ldots, \omega'(0)$ in the expression of $\widehat{\varepsilon}$, found in the described interactive procedure, we obtain

$$\widehat{\varepsilon} = \frac{1}{2}c_2\left(c_2^2 - c_3\right)\left[2c_2c_4 + 4c_2^4 - 8c_2^2c_3 + \psi''(0)\left(c_3^2 + c_2^4 - 2c_2^2c_3\right)\right]e^8 + O\left(\varepsilon^9\right). \tag{2.4}$$

Observe from (2.4) that $\psi''(0)$ must be bounded.

According to the above analysis we can state the following theorem.

Theorem 2.1. *If x_0 is a sufficiently close approximation to a zero α of f, then the family of three-point methods*

$$\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)}{f(x_k) - 2f(y_k)}, \\
x_{k+1} &= z_k - \frac{f(z_k)}{f'(x_k)\phi(t_k)\psi(s_k)\omega(v_k)}, \quad t_k = \frac{f(y_k)}{f(x_k)}, \\
s_k &= \frac{f(z_k)}{f(y_k)}, \quad v_k = \frac{f(z_k)}{f(x_k)},
\end{aligned} \tag{2.5}$$

has the order eight if sufficiently times differentiable functions ϕ, ψ, and ω are chosen so that the following conditions are fulfilled:

$$\begin{gathered}
\phi(0) = 1, \quad \phi'(0) = 2, \quad \phi''(0) = -2, \quad \phi'''(0) = 0, \\
\psi(0) = 1, \quad \psi'(0) = -1, \quad \left|\psi''(0)\right| < \infty, \quad \omega(0) = 1, \quad \omega'(0) = -2.
\end{gathered} \tag{2.6}$$

Values of higher order derivatives of ϕ, ψ, and ω, not explicitly given in (2.6)*, can be arbitrary at the point 0.*

Weight functions ϕ, ψ, and ω should be chosen as simple as possible. One of the simplest forms is that obtained by using the Taylor polynomials of these functions according to (2.6), that is,

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)}{f(x_k) - 2f(y_k)},$$

$$x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)\left[1 - 2f(y_k)/f(x_k) - \left(f(y_k)/f(x_k)\right)^2\right]\left[1 - f(z_k)/f(y_k)\right]\left[1 - 2f(z_k)/f(x_k)\right]}. \tag{2.7}$$

Using the approximation $1/(1-aq) \approx 1+aq$ for sufficiently small $|q|$, the last iterative formula may be modified to the form

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_k = y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)}{f(x_k) - 2f(y_k)}, \tag{2.8}$$
$$x_{k+1} = z_k - \frac{f(z_k)\left[1 + f(z_k)/f(y_k)\right]\left[1 + 2f(z_k)/f(x_k)\right]}{f'(x_k)\left[1 - 2f(y_k)/f(x_k) - \left(f(y_k)/f(x_k)\right)^2\right]}.$$

Some other simple forms of functions ϕ, ψ, and ω are

$$\phi(t) = \frac{1 - 4t^2(1-t)}{(1+t)^2},$$

$$\phi(t) = \frac{5 - 12t}{5 - 2t + t^2},$$

$$\psi(s) = \left(1 - \frac{s}{n}\right)^n, \quad \text{preferable } \psi(s) = \left(1 - \frac{s}{2}\right)^2 \text{for } n = 2,$$

$$\psi(s) = \frac{1 + as}{1 + (a+1)s}, \quad a \in \mathbf{R}, \text{ preferable } \psi(s) = \frac{1}{1+s} \text{ for } a = 0,$$

$$\psi(s) = \frac{1}{1 + s + cs^2}, \quad c \in \mathbf{R},$$

$$\omega(v) = \left(1 - \frac{2v}{n}\right)^n, \quad \text{preferable } \omega(v) = (1-v)^2 \text{ for } n = 2,$$

$$\omega(v) = \frac{1+bv}{1+(b+2)v}, \quad b \in \mathbf{R}, \text{ preferable } \omega(v) = \frac{1}{1+2v} \text{ for } b = 0,$$

$$\omega(v) = \frac{1}{1+2v+dv^2}, \quad d \in \mathbf{R}. \tag{2.9}$$

It is interesting to note that functions $\varphi(s) = e^{-s}$ and $\omega(v) = e^{-2v}$ do satisfy the requested conditions (2.6), but the calculation of exponential function increases computational cost, so such choice is not acceptable.

3. Numerical Results

The family of three-point methods (2.5) has been tested on numerous nonlinear equations along with some other methods of the same convergence rate. The programming package *Mathematica* with multiprecision arithmetic (800 significant decimal digits) was employed to provide very high accuracy. For comparison purposes, we have also tested the three-point methods of optimal order eight given below.

Bi-Wu-Ren's Family [4]

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = y_k - h(\mu_k)\frac{f(y_k)}{f'(x_k)}, \tag{3.1}$$

$$x_{k+1} = z_k - \frac{f(x_k) + \beta f(z_k)}{f(x_k) + (\beta - 2)f(z_k)} \cdot \frac{f(z_k)}{f[z_k, y_k] + f[z_k, x_k, x_k](z_k - y_k)} \quad (\beta \in \mathbf{R}),$$

where $\mu_k = f(y_k)/f(x_k), h(t)$ is a real-valued function and

$$f[z,y] = \frac{f(z) - f(y)}{z - y}, \qquad f[z,x,x] = \frac{f[z,x] - f'(x)}{z - x}. \tag{3.2}$$

We have tested two methods belonging to the family (3.1), obtained by choosing two different forms of the weight function h in the same way as in [4] (see Tables 1 and 2).

Kung-Traub's Method

Version 1 without derivatives [5]:

$$\begin{aligned} y_k &= x_k - \frac{\gamma f(x_k)^2}{f(x_k+\gamma f(x_k)) - f(x_k)}, \\ z_k &= y_k - \frac{f(y_k) f(x_k+\gamma f(x_k))}{(f(x_k+\gamma f(x_k)) - f(y_k)) f[x_k, y_k]}, \quad (k = 0, 1, \ldots), \\ x_{k+1} &= z_k - \frac{f(y_k) f(x_k+\gamma f(x_k))(y_k - x_k + f(x_k)/f[x_k, z_k])}{(f(y_k) - f(z_k))(f(x_k+\gamma f(x_k)) - f(z_k))} + \frac{f(y_k)}{f[y_k, z_k]}, \end{aligned} \tag{3.3}$$

where γ is a real parameter.

Kung-Traub's Method

Version 2 with derivative [5]:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= y_k - \frac{f(x_k)^2 f(y_k)}{f'(x_k)(f(x_k) - f(y_k))^2}, \quad (k = 0, 1, \ldots), \\ x_{k+1} &= z_k - \frac{f(x_k)^2 f(y_k)}{\Delta_{yz}^{(k)}} \left[\frac{1}{\Delta_{xz}^{(k)}} \left(\frac{x_k - z_k}{\Delta_{xz}^{(k)}} - \frac{1}{f'(x_k)} \right) - \frac{f(y_k)}{f'(x_k)\left(\Delta_{xy}^{(k)}\right)^2} \right], \end{aligned} \tag{3.4}$$

where, for example,

$$\Delta_{xz}^{(k)} = f(x_k) - f(z_k). \tag{3.5}$$

Liu-Wang's Method [6]

$$\begin{gathered} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \frac{f(y_k)}{f'(x_k)} H\left(\frac{f(y_k)}{f(x_k)}\right), \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)} \left[U\left(\frac{f(y_k)}{f(x_k)}\right) + V\left(\frac{f(z_k)}{f(y_k)}\right) + W\left(\frac{f(z_k)}{f(x_k)}\right) \right] \quad (a \in \mathbf{R}), \\ H(0) = 1, \qquad H'(0) = 2, \qquad U(0) = 1 - V(0) - W(0), \qquad U'(0) = 2, \\ U''(0) = 2 + H''(0), \qquad U'''(0) = -24 + 6H''(0) + H'''(0), \qquad V'(0) = 1, \qquad W'(0) = 4. \end{gathered} \tag{3.6}$$

Table 1

Methods	$f(x) = \log(x^2+1) + e^x \sin x$, $x_0 = 0.3$, $\alpha = 0$ $\lvert x_1-\alpha \rvert$	$\lvert x_2-\alpha \rvert$	$\lvert x_3-\alpha \rvert$	r_c (3.8)
New IM (2.5) $\phi = 1-2t-t^2$, $\psi = 1-s$, $\omega = 1-2v$	3.92(−4)	1.04(−25)	2.52(−198)	7.9998
new IM (2.5) $\phi = 1-2t-t^2-5t^4$, $\psi = 1-s-s^2$, $\omega = 1-2v-v^2$	8.66(−5)	1.57(−30)	1.82(−236)	7.9999
new IM (2.5) $\phi = 1-2t-t^2-5t^4$, $\psi = 1/(1+s+4s^2)$, $\omega = 1/(1+v)^2$	7.44(−5)	6.56(−31)	2.37(−239)	8.0000
Bi-Wu-Ren's IM (3.1), method 1	6.52(−5)	1.14(−32)	9.57(−255)	8.0000
Bi-Wu-Ren's IM (3.1), method 2	4.08(−4)	2.44(−25)	3.52(−195)	8.0028
Kung-Traub's IM (3.3)	8.13(−4)	2.16(−22)	5.45(−171)	7.9993
Kung-Traub's IM (3.4)	7.84(−4)	1.56(−22)	3.96(−172)	7.9993
Liu-Wang's IM (3.6)	5.74(−4)	4.59(−24)	7.81(−185)	7.9996

Table 2

Methods	$f(x) = 1 + e^{x^3-x} - \cos(1-x^2) + x^3$, $x_0 = -1.65$, $\alpha = -1$ $\lvert x_1-\alpha \rvert$	$\lvert x_2-\alpha \rvert$	$\lvert x_3-\alpha \rvert$	r_c (3.8)
New IM (2.5) $\phi = 1-2t-t^2$, $\psi = 1-s$, $\omega = 1-2v$	3.04(−5)	1.81(−37)	2.85(−295)	8.0000
new IM (2.5) $\phi = 1-2t-t^2-5t^4$, $\psi = 1-s-s^2$, $\omega = 1-2v-v^2$	2.38(−5)	3.44(−38)	6.47(−301)	8.0000
new IM (2.5) $\phi = 1-2t-t^2-5t^4$, $\psi = 1/(1+s+4s^2)$, $\omega = 1/(1+v)^2$	8.31(−6)	3.12(−41)	1.24(−324)	8.0000
Bi-Wu-Ren's IM (3.1), method 1	3.28(−5)	7.17(−38)	3.71(−299)	7.9999
Bi-Wu-Ren's IM (3.1), method 2	3.16(−5)	1.66(−37)	9.59(−296)	8.0000
Kung-Traub's IM (3.3)	9.10(−5)	2.28(−33)	3.60(−262)	8.0000
Kung-Traub's IM (3.4)	2.85(−5)	1.75(−37)	3.54(−295)	8.0000
Liu-Wang's IM (3.6)	3.64(−5)	2.59(−36)	1.73(−285)	8.0000

Remark 3.1. There are other three-point methods with optimal order eight, see, for instance, [3, 7–15]. However, these methods produce results of approximately same quality so that did not display them in Tables 1 and 2.

For demonstration, among many numerical experiments, we have selected the following two functions:

$$\begin{gathered} f(x) = \log\left(x^2+1\right) + e^x \sin x, \quad x_0 = 0.3,\ \alpha = 0, \\ f(x) = 1 + e^{x^3-x} - \cos\left(1-x^2\right) + x^3, \quad x_0 = -1.65,\ \alpha = -1. \end{gathered} \tag{3.7}$$

The errors $|x_k - \alpha|$ of approximations to the zeros are given in Tables 1 and 2, where $A(-h)$ denotes $A \times 10^{-h}$. These tables include the values of the computational order of convergence r_c calculated by the formula

$$r_c = \frac{\log\left|f(x_k)/f(x_{k-1})\right|}{\log\left|f(x_{k-1})/f(x_{k-2})\right|}, \tag{3.8}$$

taking into consideration the last three approximations in the iterative process.

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