On an Efficient Family of Simultaneous Methods for Finding Polynomial Multiple Zeros

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Abstract An iterative method for the simultaneous determination of multiple zeros of algebraic polynomials is stated. This method is more efficient compared to all existing simultaneous methods based on fixed point relations. To attain very high computational efficiency, a suitable correction resulting from Li-Liao-Cheng's two-point fourth-order method of low computational complexity is applied. The presented convergence analysis shows that the convergence rate of the basic method is increased from three to six using this special type of correction and applying only ν additional polynomial evaluations per iteration, where ν is the number of distinct zeros. Computational aspects and some numerical examples are given to demonstrate high computational efficiency and very fast convergence of the proposed method.

1 Introduction

The aim of this paper is to construct an iterative method for the simultaneous determination of all multiple zeros of a polynomial with a very high computational efficiency. Actually, the proposed method is ranked as the most efficient among existing methods in the class of simultaneous methods for approximating polynomial multiple zeros based on fixed point relations. The presented iterative formula relies on the fixed point relation of Gargantini's type [3]. A very high computational

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efficiency is attained by employing suitable corrections which enable very fast convergence (equal to six) with minimal additional computational costs. In fact, these corrections arise from Li-Liao-Cheng's two-point root-solver [4] with optimal order of convergence four. More details about multi-point methods may be found, e.g., in [8] and [10].

The paper is organized as follows. In Sect. 2 we present the improved iterative method for the simultaneous determination of polynomial multiple zeros, starting from a suitable fixed-point relation. The convergence theorem stated in Sect. 3 asserts that the convergence order of the proposed method is six. Finally, Sect. 4 contains an analysis of computational efficiency which shows that the proposed simultaneous method is the most efficient among all existing methods based on fixed point relations. In addition, two numerical examples are given to demonstrate exceptional convergence speed of the proposed method.

2 Accelerated Simultaneous Method

Let $f(z) = \prod_{j=1}^{\nu}(z-\zeta_j)^{\mu_j}$ be a monic polynomial of degree n with multiple real or complex zeros $\zeta_1, \dots, \zeta_\nu$ of respective multiplicities $\mu_1, \dots, \mu_\nu$ $(\nu \le n)$, and let

$$u(z) = \frac{f(z)}{f'(z)} = \left[\frac{d}{dz}\log f(z)\right]^{-1} = \left(\sum_{j=1}^{\nu}\frac{\mu_j}{z-\zeta_j}\right)^{-1}. \tag{1}$$

To construct an iterative method for the simultaneous determination of polynomial multiple zeros, we single out the term $z-\zeta$ from (1) and derive the following fixed point relation

$$\zeta_i = z - \frac{\mu_i}{\frac{1}{u(z)} - \sum\limits_{\substack{j\in \boldsymbol{I}_\nu \\ j\ne i}} \frac{\mu_j}{z-\zeta_j}} \quad (i \in \boldsymbol{I}_\nu := \{1,\dots,\nu\}). \tag{2}$$

This relation was used in [3] for the construction of iterative methods for the simultaneous inclusion of multiple zeros of polynomials in complex circular arithmetic.

Let $z_1, \dots, z_\nu$ be distinct approximations to the zeros $\zeta_1, \dots, \zeta_\nu$. Setting $z = z_i$ and substituting the zeros ζ_j by some approximations z_j^* in the right-hand side of (2), one obtains the following iterative method

$$\hat{z}_i = z_i - \frac{\mu_i}{\dfrac{1}{u(z_i)} - \sum\limits_{\substack{j\in \boldsymbol{I}_\nu \\ j\ne i}} \dfrac{\mu_j}{z_i - z_j^*}} \quad (i \in \boldsymbol{I}_\nu) \tag{3}$$

for the simultaneous determination of all multiple zeros of the polynomial f. Here $\hat{z}_i$ denotes a new approximation to the zero ζ_i. The choice $z_j^* = z_j$ in (3) gives the third-order method of Ehrlich-Aberth's type for multiple zeros

$$\hat{z}_i = z_i - \frac{\mu_i}{\dfrac{1}{u(z_i)} - \displaystyle\sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i - z_j}} \quad (i \in \boldsymbol{I}_\nu), \tag{4}$$

see [1, 2]. Furthermore, putting Schröder's approximations $z_j^* = z_j - \mu_j u(z_j)$ in (3), the following accelerated method of the fourth order is obtained (see [5]),

$$\hat{z}_i = z_i - \frac{\mu_i}{\dfrac{1}{u(z_i)} - \displaystyle\sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i - z_j + \mu_j u(z_j)}} \quad (i \in \boldsymbol{I}_\nu). \tag{5}$$

Note that the iterative method (5) reduces to Nourein's method [6] in the case of simple zeros.

Regarding (2)–(5), it is evident that the better approximations z_j^* give the more accurate approximations $\hat{z}_i$. Indeed, if $z_j^* \to \zeta_j$ then the right-hand side of (3) tends to ζ_i. We apply this idea to construct a higher order method.

The iterative method (5) of the fourth order is obtained using Schröder's method $z_j^* = z_j - \mu_j u(z_j)$ of the second order. Further acceleration of the convergence speed can be obtained by using methods of higher order for finding a single multiple zero. In this paper we use the following two-point method for solving nonlinear equations proposed in [4]

$$\hat{z} = z - u(z) \cdot \frac{\beta + \gamma t(z)}{1 + \delta t(z)}, \quad t(z) = \frac{f'(z - \theta u(z))}{f'(z)}, \tag{6}$$

where

$$\theta = \frac{2m}{m+2}, \quad \beta = -\frac{m^2}{2}, \quad \gamma = \frac{m(m-2)}{2}\left(\frac{m}{m+2}\right)^{-m}, \quad \delta = -\left(\frac{m}{m+2}\right)^{-m}$$

and m is the multiplicity of the wanted zero ζ of a function f (not necessarily algebraic polynomial in general). The order of convergence of the iterative method (6) is four, that is,

$$\hat{z} - \zeta = O_M\big((z - \zeta)^4\big) \tag{7}$$

holds (for the proof, see [4]). Here O_M is a symbol which points to the fact that two complex numbers w_1 and w_2 have moduli of the same order (that is, $|w_1| = O(|w_2|)$, O is the Landau symbol), written as $w_1 = O_M(w_2)$.

In the sequel, we substitute z by the approximation z_j of ζ_j and m by the corresponding multiplicity μ_j of ζ_j. The approximation z_j^* appearing in (3) is calculated by (6), that is,

$$z_j^* = z_j - u_j \cdot \frac{\beta_j + \gamma_j t_j}{1 + \delta_j t_j},$$

where we put $u_j = u(z_j),\ t_j = f'(z_j - \theta_j u_j)/f'(z_j)$ and

$$\theta_j = \frac{2\mu_j}{\mu_j + 2},\ \beta_j = -\frac{\mu_j^2}{2},\ \gamma_j = \frac{\mu_j(\mu_j - 2)}{2}\Big(\frac{\mu_j}{\mu_j + 2}\Big)^{-\mu_j},\ \delta_j = -\Big(\frac{\mu_j}{\mu_j + 2}\Big)^{-\mu_j}.$$

Now, from (3) we obtain a new method for the simultaneous approximation of all simple or multiple zeros of a given polynomial,

$$\hat{z}_i = z_i - \frac{\mu_i}{\dfrac{1}{u(z_i)} - \displaystyle\sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \dfrac{\mu_j}{z_i - z_j + u_j \cdot \frac{\beta_j + \gamma_j t_j}{1 + \delta_j t_j}}} \quad (i \in \boldsymbol{I}_\nu). \tag{8}$$

3 Convergence Theorem

Theorem 1. *If initial approximations $z_1, \ldots, z_\nu$ are sufficiently close to the distinct zeros $\zeta_1, \ldots, \zeta_\nu$ of a given polynomial, then the order of convergence of the simultaneous method (8) is six.*

Proof. Let us introduce the errors of approximations $\varepsilon_j = z_j - \zeta_j,\ \hat{\varepsilon}_j = \hat{z}_j - \zeta_j$. According to the conditions of Theorem 1, we can assume that $\varepsilon_i = O_M(\varepsilon_j)$ for any pair $i, j \in \boldsymbol{I}_\nu$. Let $\varepsilon \in \{\varepsilon_1, \ldots, \varepsilon_n\}$ be the error of maximal modulus with $\varepsilon_j = O_M(\varepsilon)$ $(j \in \boldsymbol{I}_\nu)$.

For brevity, let

$$z_j^* = z_j - u_j \cdot \frac{\beta_j + \gamma_j t_j}{1 + \delta_j t_j}, \quad d_i = \sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \frac{\mu_j (z_j^* - \zeta_j)}{(z_i - \zeta_j)(z_i - z_j^*)}.$$

Then, starting from (8) and using (1) we obtain

$$\hat{z}_i = z_i - \frac{\mu_i}{\dfrac{\mu_i}{\varepsilon_i} + \displaystyle\sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \dfrac{\mu_j}{z_i - \zeta_j} - \displaystyle\sum_{\substack{j \in \boldsymbol{I}_\nu \\ j \neq i}} \dfrac{\mu_j}{z_i - z_j^*}} = z_i - \frac{\mu_i \varepsilon_i}{\mu_i - \varepsilon_i d_i}, \tag{9}$$

and hence

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \varepsilon_i - \frac{\mu_i \varepsilon_i}{\mu_i - \varepsilon_i d_i} = \frac{-\varepsilon_i^2 d_i}{\mu_i - \varepsilon_i d_i}. \tag{10}$$

According to (7) we have $d_i = O_M(\varepsilon^4)$ and from (10) we find

$$\hat{\varepsilon} = O_M(\varepsilon^6),$$

since the denominator of (10) tends to μ_i when $\varepsilon_i \to 0$. Therefore, the order of convergence of the simultaneous method (8) is six. □

4 Computational Aspects

From a practical point of view, it is of great importance to estimate the computational efficiency of any iterative root-finding method since it is closely connected to the features such as the number of necessary numerical operations in computing the zeros with the required accuracy, the convergence speed, processor time of a computer, etc. The knowledge of the computational efficiency is of particular interest in designing a package of root-solvers. More details about this topic may be found in [7, Chap. 6].

In this section we compare the convergence behavior and computational efficiency of the methods (4), (5) and the new simultaneous method (8). This comparison procedure is entirely justified since the analysis of efficiency given in [7, Chap. 6] for several computing machines showed that the method (5) has the highest computational efficiency in the class of simultaneous methods based on fixed point relations.

Comparing the iterative formulas (5) and (8) we observe that the new formula (8) requires ν new polynomial evaluations per iterations in relation to (5). Hence we conclude that the minimal computational efficiency of the iterative method (8) appears when $\nu = n$, that is, when all zeros are simple. For this reason we will consider this "worst case" in our computational analysis. In a similar way as in [9] and several other papers in the topic, we estimated computational efficiency of the iterative methods (4), (5) and (8) using the *efficiency index* given by

$$E(\text{IM}) = \frac{\log r}{d}, \tag{11}$$

where r is the order of convergence of the iterative method (IM), and d is its computational cost. The computation cost d is evaluated using the total number of basic arithmetic operations per iteration taken with certain *weights* depending on the execution times of operations, see [9] for details.

We calculated the percent ratios

$$\rho_{8,4}(n) = (E((8),n)/E((4),n) - 1) \cdot 100 \ \ (\text{in } \%), \qquad \text{(F/EA \%)}$$

$$\rho_{8,5}(n) = (E((8),n)/E((5),n) - 1) \cdot 100 \ \ (\text{ in } \%), \qquad \text{(F/N\%)}$$

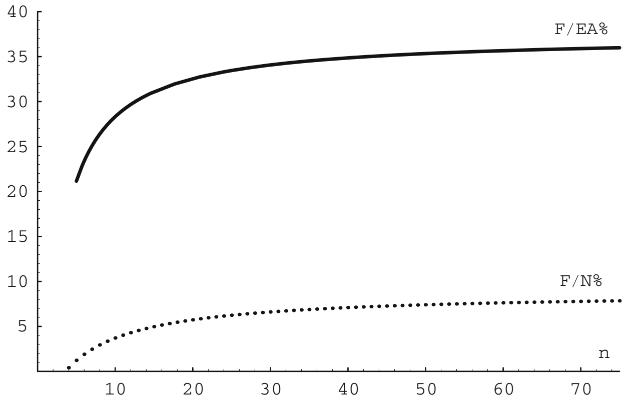

Fig. 1 Ratios of efficiency indices

where EA, N and F stand for the method (4) of Ehrlich-Aberth's type, the method (5) of Nourein's type and the new method (8), respectively. These ratios are graphically displayed in Fig. 1 as the functions of the polynomial degree n and show the (percent) improvement of computational efficiency of the new method (8) in relation to the methods (4) and (5). In Fig. 1 $\rho_{8,4}(n)$ is drawn by full line and $\rho_{8,5}(n)$ by dotted line.

It is evident from Fig. 1 that the new method (8) is more efficient than the methods (4) and (5). The improvement is especially expressive in regard to the method (4) of Ehrlich-Aberth's type (F/EA % – full line). Having in mind the mentioned fact on the dominant efficiency of the Nourein-like method, it follows that the proposed family of simultaneous methods (8) is the *most efficient method* for the simultaneous determination of polynomial multiple zeros in the class of methods based on fixed point relations.

To demonstrate the convergence behavior of the methods (4), (5) and (8), we tested a number of polynomial equations; for illustration, among a number of tested algebraic polynomials we selected two examples. To present the results of the third iteration, we applied the computational software package *Mathematica* with multiple-precision arithmetic.

As a measure of accuracy of the obtained approximations, we calculated Euclid's norm

$$e^{(m)} := ||\boldsymbol{z}^{(m)} - \boldsymbol{\zeta}||_2 = \left(\sum_{i=1}^{\nu} \left| z_i^{(m)} - \zeta_i \right|^2 \right)^{1/2} \quad (m = 0, 1, \ldots),$$

where $\boldsymbol{z}^{(m)} = \left(z_1^{(m)}, \ldots, z_\nu^{(m)}\right)$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_\nu)$.

Table 1 Euclid's norm of the errors – Example 1

Methods →	(4)	(5)	(8)
$e^{(1)}$	2.81(−1)	1.62(−1)	1.80(−1)
$e^{(2)}$	2.61(−3)	6.00(−5)	9.03(−7)
$e^{(3)}$	2.93(−9)	1.92(−18)	1.21(−39)

Table 2 Euclid's norm of the errors – Example 2

Methods →	(4)	(5)	(8)
$e^{(1)}$	1.10(−1)	5.57(−2)	2.18(−2)
$e^{(2)}$	7.24(−5)	2.38(−7)	7.44(−13)
$e^{(3)}$	1.64(−14)	3.34(−29)	3.65(−75)

Example 1. Methods (4), (5) and (8) were applied for the simultaneous approximation to the zeros of the polynomial

$$f_{18}(z) = (z+1)^2(z+2)^3(z^2-2z+2)^2(z^2+1)^2(z-2)^3(z+2-i)^2.$$

The following starting approximations were selected ($e^{(0)} \approx 1.50$)

$$\begin{aligned} &z_1^{(0)} = -1.3+0.2i,\ z_2^{(0)} = -2.2-0.3i,\ z_3^{(0)} = 1.3+1.2i,\ z_4^{(0)} = 0.7-1.2i,\\ &z_5^{(0)} = -0.2+0.8i,\ z_6^{(0)} = 0.2-1.3i,\quad z_7^{(0)} = 2.2-0.3i,\ z_8^{(0)} = -2.2+0.7i. \end{aligned}$$

The entries of the maximal errors obtained in the first three iterations are given in Table 1.

Example 2. In order to find the zeros of the polynomial

$$f_{20}(z) = (z+1)^2(z+3)^3(z^2-2z+2)^2(z-1)^3(z^2-4z+5)^2(z^2+4z+5)^2,$$

we applied the same methods. The starting approximations were ($e^{(0)} \approx 1.43$)

$$\begin{aligned} &z_1^{(0)} = -1.3+0.2i,\ z_2^{(0)} = -2.8-0.2i,\ z_3^{(0)} = 1.2+1.3i,\\ &z_4^{(0)} = 0.8-1.2i,\quad z_5^{(0)} = 0.8-0.3i,\quad z_6^{(0)} = -1.8+1.2i,\\ &z_7^{(0)} = -1.8-1.2i,\ z_8^{(0)} = 1.8+0.8i,\quad z_9^{(0)} = 1.8-1.2i. \end{aligned}$$

The entries of the maximal errors obtained in the first three iterations are given in Table 2.

From Tables 1 and 2 and a number of tested polynomial equations we can conclude that the proposed family (8) produces approximations of considerably great accuracy; two iterative steps are usually sufficient in solving most practical problems when initial approximations are reasonably close to the zeros.

The presented analysis of computational efficiency shows that the family (8) is more efficient than all existing methods for multiple zeros based on fixed point relations.

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