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On a general transformation of multipoint root-solvers

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Optimal multipoint methods for solving nonlinear equations of arbitrary order of convergence are investigated. A low cost transformation that converts Newton-preconditioned methods into a derivative free variant is presented. This transforming procedure preserves both algorithm body structure and order of convergence of the original scheme. Another useful application of the proposed transformation is the acceleration of convergence order of non-optimal methods.

1. Introduction

An extensive study of multipoint methods for solving nonlinear equations has been triggered in the last decade. The increased interest for this type of iterative methods lies in dominant computational efficiency of multipoint methods relative to classical one-point methods such as Newton's, Halley's, Laguerre's and Ostrowski's square-root method. Investigations in this area (see e.g. [1–3]) led to a set of instructions for the best use of information in order to achieve maximum accuracy of the obtained approximations keeping the fixed computational cost. Such class of methods is referred to as *optimal* and it is designed in two main streams: Newton based methods and Traub–Steffensen based derivative free methods.

In certain situations the existing Newton based algorithm has to be modified into a derivative free scheme. The presence of singularity or a too expensive derivative evaluation adds to such examples. In these cases, in order to remove defects in the applied algorithm, we are able to propose an elegant and easy way to perform suitable transformation, which is the main goal of this paper. Instead of searching for a new zero-approximation, or designing of an entirely new subroutine, we give an 'easy way out'. A low cost derivative transformation, denoted by $\mathcal{T}$, allows the mentioned modification preserving both the optimal order of convergence and unchanged algorithm body structure.

In the papers [4–6], etc. authors recognized the usefulness of the transformation $\mathcal{T}$ on several particular multipoint methods. In this paper we generalize their results to any multipoint method. Major part of this communication is dedicated to obtaining sufficient conditions under which the transformation $\mathcal{T}$ gives optimal results. These conditions are derived from the application of $\mathcal{T}$ on particular members of the general Kung–Traub family derived and studied in the fundamental paper [7]. Using Traub's results [1, Theorem 2–9] on the representation of iteration functions (IF for short), we expand the

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transformation $\mathcal{T}$ from Kung–Traub's IF to every other optimal method with the same type of the informational set: one derivative and two or more function evaluations (FE for short).

We will show that the transformation $\mathcal{T}$ is a powerful tool for obtaining entirely new derivative free families of iteration functions. We emphasize that the main contribution of this paper is the transforming procedure $\mathcal{T}$ and its application to a wide range of methods. Constructing new multipoint methods and explicit determination of their asymptotic error constants are of secondary interest.

We will be analyzing error relations of *stationary* multipoint methods without memory and their modifications. Investigated iterative procedures serve for approximating a simple root α of a nonlinear equation of the form

$$f(x) = 0.$$

For simplicity, we omit iteration indices in formulas involved, and keep error track only of two successive approximations x and $\widehat{x}$, where $\widehat{x} = \phi(x)$ is presumably an improved approximation to the zero α obtained by an IF ϕ. For these methods a simple definition of order of convergence suffices. Moreover, such methods have integral order of convergence [1,2].

Definition 1. Let ϕ be an IF. If there exist $r \in \mathbb{R}$ and a nonzero constant A_r such that

$$\lim_{x\to\alpha} \frac{\phi(x)-\alpha}{(x-\alpha)^r} = A_r,$$

then $r = r(\phi)$ is the order of convergence of IF ϕ.

We will use symbols $\sim$, o and $\mathcal{O}$ according to the convention:

- If $\lim_{x\to a} \frac{f}{g} = C < \infty,\ C \neq 0$, we shall write $f = \mathcal{O}(g)$ or $f \sim Cg,\ x \to a$.
- If $\frac{f}{g} \to 0$ when $x \to a$, then we denote $f = o(g),\ x \to a$.

In this paper we will use the well-known Kung–Traub conjecture [7].

Kung–Traub conjecture. n-point iterative methods without memory requiring $n + 1$ function evaluations, have order of convergence at most 2^n.

A class of iterative methods without memory that uses $n + 1$ FE to obtain the maximal order 2^n is called *optimal*.

The paper is organized as follows. In Sections 2 and 3 the transformation $\mathcal{T}$ is applied to two-point and three-point methods belonging to Kung–Traub's family [7]. These two iterative methods are particular members of the general n-point family based on inverse interpolation and Newton's IF. Its members will be denoted by $\mathcal{R}_n$ in the sequel. The other family defined in [7] is derivative free and based on inverse interpolation. Here it will be denoted by $\mathcal{P}_n$. Error estimate of the interpolating polynomial and divided differences properties play the crucial role in final assessments. Conclusions of the second and third section are unified and generalized on Newton's type n-point optimal methods in Theorem 1, Section 4. In Section 5, we explore further application of $\mathcal{T}$ to the acceleration of non-optimal methods. In Section 6 we give several examples of two-, three- and four-point Newton based methods and their $\mathcal{T}$-modifications. The end of Section 6 contains results of numerical tests of the considered methods. Numerical examples confirm theoretical results given in earlier sections.

2. One-point and two-point methods

To eliminate derivative evaluation from an iterative procedure with Newton's method in the first step

$$\mathcal{N}(x) := x - \frac{f(x)}{f'(x)},$$

we introduce a convenient transformation $\mathcal{T}$. Our approach relies on a substitute FE in such a way that it preserves both the structure and the order of convergence of the original entry scheme.

Derivative estimate by the divided difference,

$$f'(x) \approx f[x, w] = \frac{f(w) - f(x)}{w - x} = \frac{f(x + \gamma f(x)) - f(x)}{\gamma f(x)} \quad (w = x + \gamma f(x))$$

is one of the most known and commonly used. The approximation $w = x+\gamma f(x)$ with a nonzero constant γ was introduced in the literature as a part of Traub–Steffensen's method [8,1]

$$\widehat{x} = \mathcal{S}(x) := x - \frac{f(x)}{f[w, x]} \equiv \mathcal{P}_1(0), \tag{1}$$

where $\mathcal{P}_1(t) = \mathcal{P}_1(t; w, x)$ is Newton's inverse interpolating polynomial of the first degree based on the information $f(x)$ and $f(w)$, that is,

$$\mathcal{P}_1(f(x)) = x, \qquad \mathcal{P}_1(f(w)) = w.$$

Remark 1. In continuation, we use similar notation for n-point methods $\mathcal{R}_n$, $\mathcal{P}_n$ and adequate inverse interpolating polynomials of degree n, $\mathcal{R}_n(t)$, $\mathcal{P}_n(t)$, without possibility of any confusion. The reason for such notation choice lies in the fact that these polynomials define individual steps of IF $\mathcal{R}_n$ and $\mathcal{P}_n$, stated by Kung and Traub in [7].

To establish the transforming procedure $\mathcal{T}$, we propose a conveniently chosen approximation $w = x + \gamma f(x)$, where $\gamma = \gamma(x)$ is not necessarily a constant and does not rely on information from the previous iteration (as it was done in many papers, see [9] and [10] for example). The transformation $\mathcal{T}$ is based on the substitution

$$f'(x) \overset{\mathcal{T}}{\mapsto} f[w, x], \quad w = x + \gamma(x)f(x) \tag{2}$$

and does not change the rest of the IF body. Computer implementation of $\mathcal{T}$ is performed by adding a single command line at the beginning of the iteration loop, possibly as part of `IF-THEN-ELSE` command,

```
w=x+g(x)*f(x); f'(x):=(f(w)-f(x))/(w-x);
```

Obviously, Newton's iteration becomes Traub–Steffensen's method (1), that is, with $w = x+\gamma f(x)$, we have $\mathcal{T}\big(\mathcal{N}(x)\big) = \mathcal{S}(x)$, and order two is preserved for $\gamma \in \mathbb{R},\ \gamma \neq 0$.

In the analysis and application of $\mathcal{T}$-transformation to multipoint methods, we begin with $\mathcal{R}_2$, the optimal two-point Kung–Traub scheme [7] based on Newton's method in the first step

$$\mathcal{R}_2 : \begin{cases} y_1 = \mathcal{N}(x) = x - \dfrac{f(x)}{f'(x)}, \\ \widehat{x} = y_1 - \dfrac{f(x)^2}{\big(f(y_1) - f(x)\big)^2} \dfrac{f(y_1)}{f'(x)}. \end{cases} \tag{3}$$

Lemma 1. *If $w = x + f(x)^m,\ m \geq 2$, then $r\big(\mathcal{T}(\mathcal{R}_2)\big) = 4$.*

Proof. Let $\mathcal{F} \equiv f^{-1}$ be an inverse function defined in some neighborhood of the sought zero α of f. Equivalent form of (3) reads

$$\begin{cases} y_1 = \mathcal{N}(x) \equiv \mathcal{R}_1(0), \\ \widehat{x} = \mathcal{R}_2(0) = y_1 + \mathcal{F}[f(x), f(x), f(y_1)]f(x)^2. \end{cases} \tag{4}$$

Here $\mathcal{R}_1(t) = \mathcal{R}_1(t; x, x)$ and $\mathcal{R}_2(t) = \mathcal{R}_2(t; x, x, y_1)$ are the inverse Hermite interpolating polynomials of degree 1 and 2, based on available information $f(x), f'(x), f(y_1)$. The polynomial $\mathcal{R}_1(t)$ satisfies interpolating conditions

$$\mathcal{R}_1(f(x)) = x, \qquad \mathcal{R}_1'(f(x)) = 1/f'(x),$$

whereas $\mathcal{R}_2(t)$ satisfies $\mathcal{R}_2(f(y_1)) = y_1$ in addition.

Let us now apply $\mathcal{T}$ to the algorithm (3). The original scheme (3) is transformed into a derivative free iteration of the same body structure

$$\begin{cases} \mathcal{T}(y_1) = z_1 = x - \dfrac{f(x)}{f[w, x]} = \mathcal{S}(x), \\ \mathcal{T}(\widehat{x}) = \widetilde{x} = z_1 - \dfrac{f(x)^2}{\big(f(z_1) - f(x)\big)^2} \dfrac{f(z_1)}{f[w, x]}. \end{cases} \tag{5}$$

It remains to determine $\gamma = \gamma(x)$ which would provide the order four of the derivative free method (5). For this task the equivalent form (4), rather than (3), becomes more suitable. Transformation $\mathcal{T}$ applied to divided difference yields

$$\begin{aligned} \mathcal{F}[f(x), f(x), f(y_1)] &= \frac{\mathcal{F}[f(x), f(y_1)] - \mathcal{F}[f(x), f(x)]}{f(y_1) - f(x)} \\ \overset{\mathcal{T}}{\longmapsto}\ \frac{\mathcal{F}[f(x), f(z_1)] - \mathcal{F}[f(w), f(x)]}{f(z_1) - f(x)} &= \mathcal{F}[f(w), f(x), f(z_1)]\frac{f(z_1) - f(w)}{f(z_1) - f(x)}, \end{aligned}$$

rewritten briefly as

$$\mathcal{T}\Big(\mathcal{F}[f(x), f(x), f(y_1)]\Big) = \mathcal{F}[f(w), f(x), f(z_1)]\big(1 - K(z_1)\big), \tag{6}$$

where $K(z) = \frac{f(w)-f(x)}{f(z)-f(x)}$. Thus, the iterative scheme (5) has an equivalent form (using divided differences)

$$\begin{cases} w = x + \gamma f(x), \qquad \mathcal{T}(y_1) = z_1 = \mathcal{P}_1(0), \\ \mathcal{T}(\widehat{x}) = \widetilde{x} = z_1 + \mathcal{F}[f(w), f(x), f(z_1)]f(x)^2\big(1 - K(z_1)\big). \end{cases} \tag{7}$$

Modifying procedure $\mathcal{T}$ applied to a particular algorithmic scheme preserves original program body. However, the obtained approximations to the zero α are not preserved. For this reason, during the error analysis, we use different symbols (z instead of y, and $\widetilde{x}$ as to $\widehat{x}$) to emphasize the impact of $\mathcal{T}$ on calculation of each individual step in the original algorithm (3).

From (7) it follows

$$\widetilde{x} = \mathcal{P}_2(0) - \mathcal{F}[f(w), f(x), f(z_1)]f(x)f(z_1)K(z_1), \tag{8}$$

where $\mathcal{P}_2(t) = \mathcal{P}_2(t;\, w, x, z_1)$ is Newton's inverse interpolating polynomial of second degree satisfying the conditions

$$\mathcal{P}_2(f(w)) = w, \qquad \mathcal{P}_2(f(x)) = x, \qquad \mathcal{P}_2(f(z_1)) = z_1.$$

Introduce the errors

$$\varepsilon = x - \alpha, \qquad \widetilde{\varepsilon} = \widetilde{x} - \alpha, \qquad \varepsilon_{z_k} = z_k - \alpha, \quad k \in \mathbb{N},$$

and abbreviations

$$C_n(x) = \frac{f^{(n)}(x)}{n!f'(x)}, \qquad c_n = C_n(\alpha) = \frac{f^{(n)}(\alpha)}{n!f'(\alpha)}, \quad n \in \mathbb{N}.$$

From the error analysis of $\mathcal{P}_n(t)$ presented in [1] and [3], it follows

$$\mathcal{P}_2(0) - \alpha = \mathcal{O}(\varepsilon\varepsilon_w\varepsilon_{z_1}) = \mathcal{O}\big(\varepsilon^4\big). \tag{9}$$

The transformed scheme (5) remains optimal (of order 4) if the difference term in (8) is

$$\mathcal{F}[f(w), f(x), f(z_1)]f(x)f(z_1)K(z_1) = \mathcal{O}(\varepsilon^k) \quad \text{for } k \geq 4, \text{ as } x \to \alpha.$$

Regarding this demand, we will choose γ in $w = x + \gamma f(x)$.

Having in mind (9) and relations

$$f(x) = \mathcal{O}(\varepsilon), \qquad f(z_1) = \mathcal{O}(\varepsilon_{z_1}) = \mathcal{O}(\varepsilon^2), \tag{10}$$

$$f(x)f(z_1)K(z_1) = \gamma\mathcal{O}\big(\varepsilon^3\big), \tag{11}$$

$$\mathcal{F}[f(w), f(x), f(z_1)] = \frac{\mathcal{F}^{(2)}(0)}{2!} + \mathcal{O}(\varepsilon)$$

and the fact that $\mathcal{F}^{(2)}(0)$ is a constant expression in $f'(\alpha)$ and c_2 (see [1, (5–12)]), it follows

$$\widetilde{\varepsilon} = \mathcal{O}(\varepsilon^4) + \gamma\mathcal{O}\big(\varepsilon^3\big).$$

According to this we conclude that

$$\gamma = \gamma(x) = \mathcal{O}(\varepsilon) = \mathcal{O}\big(f(x)\big)$$

secures the optimality for (5) in the sense of keeping the same optimal order of convergence. □

Computationally non-expensive choice for the auxiliary approximation w is $w = x + f(x)^m$ where $m \geq 2$.

3. Three-point IF

Considering three-point root-solvers, as an entry of the proposed transformation $\mathcal{T}$ we choose $\mathcal{R}_3$, the three-point member of the Kung–Traub generalized family $\mathcal{R}_n$.

The three-point iterative scheme $\mathcal{R}_3$ reads

$$\begin{cases} y_1 = \mathcal{N}(x), \\ y_2 = y_1 - \dfrac{f(x)^2}{\big(f(y_1)-f(x)\big)^2}\dfrac{f(y_1)}{f'(x)}, \\ \widehat{x} = y_2 - f(x)^2 f(y_1)\dfrac{f(x)^2+f(y_1)\big(f(y_1)-f(y_2)\big)}{\big(f(y_1)-f(x)\big)^2\big(f(y_2)-f(x)\big)^2\big(f(y_1)-f(y_2)\big)}\dfrac{f(y_2)}{f'(x)}. \end{cases} \tag{12}$$

The method (12) is of optimal order 8 and it requires one derivative and three function evaluations.

Lemma 2. *If* $w = x + f(x)^m,\ m \geq 3$, *then* $r\big(\mathcal{T}(\mathcal{R}_3)\big) = 8$.

Proof. After applying $\mathcal{T}$ to (12), the following derivative free variant is obtained,

$$\begin{cases} z_1 = \mathcal{S}(x), \\ z_2 = z_1 - \dfrac{f(x)^2}{\big(f(z_1)-f(x)\big)^2}\dfrac{f(z_1)}{f[w,x]}, \\ \widetilde{x} = z_2 - f(x)^2 f(z_1)\dfrac{f(x)^2+f(z_1)\big(f(z_1)-f(z_2)\big)}{\big(f(z_1)-f(x)\big)^2\big(f(z_2)-f(x)\big)^2\big(f(z_1)-f(z_2)\big)}\dfrac{f(z_2)}{f[w,x]}. \end{cases} \tag{13}$$

Since the program flow chart of IF (12) remains unaltered by the transformation $\mathcal{T}$, we will concentrate on the question how $\mathcal{T}$ affects the obtained approximations of each particular step $\mathcal{T}(y_1) = z_1,\ \mathcal{T}(y_2) = z_2$ and $\mathcal{T}(\widehat{x}) = \widetilde{x}$.

A more compact, equivalent form of (12) reads

$$\begin{cases} y_1 = \mathcal{N}(x), \\ y_2 = \mathcal{R}_2(0), \\ \widehat{x} = \mathcal{R}_3(0) = y_2 - \mathcal{F}[f(x), f(x), f(y_1), f(y_2)]f(x)^2 f(y_1). \end{cases} \tag{14}$$

The results of applying $\mathcal{T}$ to the scheme (4), expressed in (6)–(11), are obviously valid for the first two steps of (14). Using the definition of higher order divided differences, we obtain

$$\mathcal{T}\Big(\mathcal{F}[f(x), f(x), f(y_1), f(y_2)]\Big) = \mathcal{F}[f(w), f(x), f(z_1), f(z_2)]\big(1-K(z_2)\big) + \mathcal{F}[f(w), f(x), f(z_1)]K(z_2, z_1), \tag{15}$$

where

$$K(z_2, z_1) = \frac{K(z_2)}{f(z_1)-f(x)} = \frac{f(w)-f(x)}{\big(f(z_2)-f(x)\big)\big(f(z_1)-f(x)\big)}.$$

Let $\mathcal{P}_3(t) = \mathcal{P}_3(t; w, x, z_1, z_2)$ denote Newton's inverse interpolating polynomial of degree 3, satisfying

$$\mathcal{P}_3(f(w)) = w, \qquad \mathcal{P}_3(f(x)) = x, \qquad \mathcal{P}_3(f(z_1)) = z_1, \qquad \mathcal{P}_3(f(z_2)) = z_2.$$

As derived in [1] and [3], the following error relation is valid

$$\mathcal{P}_3(0) - \alpha = \mathcal{O}(\varepsilon_w \varepsilon \varepsilon_{z_1} \varepsilon_{z_2}).$$

After some tedious but elementary transformation we conclude that $\mathcal{T}$, applied to (12) or (14), produces approximations in the following manner

$$\begin{cases} z_1 = \mathcal{S}(x), \\ z_2 = \mathcal{P}_2(0) - \mathcal{F}[f(w), f(x), f(z_1)]f(x)f(z_1)K(z_1), \\ \widetilde{x} = \mathcal{P}_3(0) + \mathcal{F}[f(w), f(x), f(z_1), f(z_2)]f(x)f(z_1)f(z_2)K(z_2) \\ \quad - \mathcal{F}[f(w), f(x), f(z_1)]f(x)f(z_1)f(z_2)K(z_2, z_1), \end{cases} \tag{16}$$

giving the error estimates

$$\begin{aligned} &\varepsilon_{z_1} = \mathcal{O}(\varepsilon^2), \qquad \varepsilon_{z_2} = \mathcal{O}(\varepsilon^4) + \mathcal{O}(\varepsilon^{m+2}), \\ &\widetilde{\varepsilon} = \mathcal{O}(\varepsilon^8) + \mathcal{O}(\varepsilon^{m+5}) + \mathcal{O}\big(\varepsilon^{2m+3}\big), \end{aligned} \tag{17}$$

based on (7), (8), (15) and the fact that $\mathcal{F}[f(w), f(x), f(z_1), f(z_2)]$ and $\mathcal{F}[f(w), f(x), f(z_1)]$ tend to some constant values in the neighborhood of α (see [1, formula (5–12)]).

Note that if we immediately assume that $\varepsilon_{z_2} = \mathcal{O}(\varepsilon^4)$ with $m \geq 2,\ w = x + f(x)^m$, that is, z_2 is an optimal sub-step of (14), then the error relations (17) read

$$\begin{aligned} &\varepsilon_{z_1} = \mathcal{O}(\varepsilon^2), \qquad \varepsilon_{z_2} = \mathcal{O}(\varepsilon^4), \\ &\widetilde{\varepsilon} = \mathcal{O}(\varepsilon^8) + \mathcal{O}\big(\varepsilon^{m+5}\big). \end{aligned}$$

From (17) it follows: the choice $w = x + f(x)^m$ where $m \geq 3$ secures the optimality of the transformed iterative scheme (13). □

Assertions for both two- and three-point IF regarding transformation $\mathcal{T}$ will be unified and generalized in the form of a theorem in the next section.

4. General *n*-point case

The entry for the transformation $\mathcal{T}$ will be $\mathcal{R}_{n+1}$, the nth member of Kung–Traub's family of order 2^{n+1} [7]

$$\begin{cases} y_1 = \mathcal{R}_1(0), \qquad y_2 = \mathcal{R}_2(0), \\ \vdots \\ \widehat{x} = \mathcal{R}_{n+1}(0) = y_n + (-1)^{n+1}\mathcal{F}[f(x), f(x), f(y_1), \ldots, f(y_n)]f(x)^2 f(y_1)\ldots f(y_{n-1}), \end{cases} \tag{18}$$

where $\mathcal{R}_k(t) = \mathcal{R}_k(t; x, x, y_1, \ldots, y_{k-1})$ is Hermite's inverse interpolating polynomial of degree k that satisfies

$$\mathcal{R}_k(f(x)) = x, \qquad \mathcal{R}_k'(f(x)) = 1/f'(x), \qquad \mathcal{R}_k(f(y_j)) = y_j, \quad j = 1, \ldots, k-1.$$

Lemma 3. *If* $w = x + f(x)^m$, $m \geq n$, *then* $r\big(\mathcal{T}(\mathcal{R}_n)\big) = 2^n$, $n \in \mathbb{N}$.

Proof. Our proof relies on mathematical induction and the fact that divided difference $\mathcal{F}[f(w), f(x), f(z_1), \ldots, f(z_k)]$ ($k \in \mathbb{N}$) tends to a constant expression in $f'(\alpha), c_2, \ldots, c_{k+1}$ in the neighborhood of α, see [1, formula (5–12)].

Lemmas 1 and 2 provide initial assertions for the inductive hypotheses: it is assumed that the auxiliary approximation $w = x + f(x)^m$, $m \geq n$ is taken and that it provides optimality for $\mathcal{T}(y_k) = z_k$, $k = 1, \ldots, n$. In other words, the error relations obtained are $\varepsilon_{z_k} = \mathcal{O}(\varepsilon^{2^k})$, $k = 1, \ldots, n$.

Observe that

$$\mathcal{F}[f(x), f(x), f(y_1), \ldots, f(y_n)] \overset{\mathcal{T}}{\longmapsto} \mathcal{F}[f(w), f(x), f(z_1), \ldots, f(z_n)]\big(1 - K(z_n)\big) + \sum_{k=1}^{n-1}(-1)^{k+1}\mathcal{F}[f(w), f(x), f(z_1), \ldots, f(z_{n-k})]K(z_n, z_{n-1}, \ldots, z_{n-k}), \tag{19}$$

where

$$K(z_n, \ldots, z_{n-k}) = \frac{K(z_n, \ldots, z_{n-k+1})}{f(z_{n-k}) - f(x)}, \qquad k \leq n-1.$$

The relation (19) is easily verified by the induction based on (6) and (15). In the same manner, (8) and (16) provide

$$\widehat{x} \overset{\mathcal{T}}{\longmapsto} \widetilde{x} = \mathcal{P}_{n+1}(0) + (-1)^n f(x) f(z_1)\ldots f(z_n) \times \sum_{k=0}^{n-1}(-1)^k \mathcal{F}[f(w), f(x), f(z_1), \ldots, f(z_{n-k})]K(z_n, \ldots, z_{n-k}), \tag{20}$$

where $\mathcal{P}_{n+1}(t) = \mathcal{P}_{n+1}(t; w, x, z_1, \ldots, z_n)$ is the inverse interpolating polynomial satisfying

$$\mathcal{P}_{n+1}(f(w)) = w, \qquad \mathcal{P}_{n+1}(f(x)) = x, \qquad \mathcal{P}_{n+1}(f(z_k)) = z_k, \quad k = 1, \ldots, n.$$

Again, according to the results given in [1] and [3] we have a general error relation

$$\mathcal{P}_{n+1}(0) - \alpha = \mathcal{O}(\varepsilon_w \varepsilon \varepsilon_{z_1} \ldots \varepsilon_{z_n}) = \mathcal{O}\big(\varepsilon^{2^{n+1}}\big).$$

Thus, the error of the transformed approximation $\widetilde{x}$ reads

$$\begin{aligned} \widetilde{\varepsilon} &= \mathcal{O}\big(\varepsilon^{2^{n+1}}\big) + \mathcal{O}\big(\varepsilon^{2^{n+1}-1}\big) \cdot \sum_{k=0}^{n-1}\mathcal{O}\big(\varepsilon^{m-k-1}\big) \\ &= \mathcal{O}\big(\varepsilon^{2^{n+1}}\big) + \mathcal{O}\big(\varepsilon^{2^{n+1}+m-(n+1)}\big) \cdot \sum_{k=0}^{n-1}\mathcal{O}\big(\varepsilon^{k}\big) \\ &= \mathcal{O}\big(\varepsilon^{2^{n+1}}\big) + \mathcal{O}\big(\varepsilon^{2^{n+1}+m-(n+1)}\big). \end{aligned}$$

Then the condition $m \geq n+1$ is sufficient to give optimal order 2^{n+1} of $\mathcal{T}\big(\mathcal{R}_{n+1}\big)$. □

The main result of this communication concerns the $\mathcal{T}$-transformation of any optimal multipoint method with Newton's pre-conditioner.

Theorem 1. *Assume that f is a function sufficiently smooth in a neighborhood of its simple zero α. Let $\phi(x)$ be an optimal multi step IF of order $r(\phi) = 2^n$ based on Newton's pre-conditioner, which consumes n evaluations of f and one evaluation of f'. Then, for x sufficiently close to α,*

$$r\big(\mathcal{T}(\phi)\big) = 2^n$$

for $w = x + f(x)^m$, $m \geq n$.

Proof. Let $u = u(x) = f(x)/f'(x)$ be Newton's correction. With regard to Theorems 2–9 from [1], if two IF $\phi_1(x)$ and $\phi_2(x)$ are of the same order of convergence r, then

$$\phi_2(x) = \phi_1(x) + \mathcal{O}\big(u(x)^p\big), \quad p \geq r \text{ when } x \to \alpha. \tag{21}$$

According to (21) we have

$$\phi = \mathcal{R}_n + \mathcal{O}\big(u^p\big), \quad p \geq 2^n.$$

Having in mind (2), the definition of $\mathcal{T}$ and the fact that $\mathcal{T}$ affects only the derivative $f'(x)$ in the input expression, the following is valid

$$\mathcal{T}\big(\phi\big) = \mathcal{T}\big(\mathcal{R}_n\big) + \mathcal{T}\big(\mathcal{O}(u^p)\big), \quad p \geq 2^n. \tag{22}$$

The assertion of Theorem 1 is then the consequence of the relation

$$\mathcal{T}\big(u(x)^p\big) = (f(x)/f[w,x])^p \sim u(x)^p. \tag{23}$$

Indeed, combining (22) and (23) gives

$$\mathcal{T}\big(\phi\big) = \mathcal{T}\big(\mathcal{R}_n\big) + \mathcal{O}(u^p), \quad p \geq 2^n,$$

meaning $r(\mathcal{T}(\phi)) \geq 2^n$.

Note that $r(\mathcal{T}(\phi)) \leq 2^n$ according to Kung–Traub's hypothesis given in Section 1, which holds for the set of information considered in this paper, see Woźniakowski [11] for the proof. With this we conclude $r\big(\mathcal{T}(\phi)\big) = r\big(\mathcal{T}(\mathcal{R}_n)\big) = 2^n$. □

In regard to the previously presented facts, the obtained results do not restrict our choice of IF to $\mathcal{R}_n$ alone. Theorem 1 suggests that $\mathcal{T}$ can be used on any member of very rich families of IF proposed in [12–14], etc. The beginning of this section could just as well be oriented to a general optimal n-point scheme presented in [15]. Each of the cited IF is based on Newton's first step, $n + 1$ FE and with the optimal order of convergence 2^n, $n \in \mathbb{N}$. For its historical merit and influence, as well for its high efficiency, we focused our investigation on the members of the Kung–Traub general family $\mathcal{R}_n$. The properties of interpolating polynomials are another reason for such a choice. The use of $\mathcal{R}_n(t)$ and $\mathcal{P}_n(t)$ provides more evident and elegant proofs. One more advantage of such approach lies in the formulas (6) and (15) which are further used to establish inductive hypotheses for statements required at the beginning of this section.

5. Further applications—acceleration of non-optimal methods

In this section we intend to point to another advantageous application of the proposed auxiliary approximation $w = x + f(x)^m$: the acceleration of non-optimal methods. Since two- and three-point iterative methods are of most interest from a practical point of view, we shall be concentrating on IF of this type. We start with a theorem presented in [16].

Theorem 2. *Let $(t, u) \mapsto h(t, u)$ be a sufficiently differentiable function of two variables in the neighborhood of the point $(0, 0)$. If*

$$h(0,0) = h_t(0,0) = h_u(0,0) = 1 \tag{24}$$

and $\gamma \in \mathbb{R} \setminus \{0\}$, then two-point IF defined by

$$\begin{cases} w = x + \gamma f(x), \qquad y = x - \dfrac{f(x)}{f[w,x]}, \\ \widehat{x} = y - h\left(\dfrac{f(y)}{f(x)}, \dfrac{f(y)}{f(w)}\right)\dfrac{f(y)}{f[w,x]}, \end{cases} \tag{25}$$

is of optimal order four.

Note that the two-point family (25) is rather general and produces a number of particular two-point methods, see [2, Ch. 2].

In the next theorem, we shall see that the optimality conditions are relaxed in the case of application of the approximation $w = x + f(x)^2$ since the number of variables in the weight function h is decreased.

Theorem 3. *If* $h(0) = 1$ *and* $h'(0) = 2$, *then the two-point family*

$$\begin{cases} w = x + f(x)^2, \qquad y = x - \dfrac{f(x)}{f[w,x]}, \\ \widehat{x} = y - h\left(\dfrac{f(y)}{f(x)}\right)\dfrac{f(y)}{f[w,x]}, \end{cases}$$

is of optimal order four.

Proof. Using symbolic computation and the following code written in the computational software package *Mathematica*, we obtain a computer based proof. We use the abbreviations:

fx $= f(x)$, fy $= f(y)$, fw $= f(w)$, fwx $= f[w,x]$, $c_k = (f^{(k)}(\alpha))/(k!f'(\alpha))$
e $= x - \alpha$, ew $= w - \alpha$, ey $= y - \alpha$, t $= f(y)/f(x)$, e1 $= \widehat{x} - \alpha$.

Then

```
In[1]:=fx=f1a(e+c2e^2+c3e^3+c4e^4+c5e^5); ew=e+fx^2;
fw=fx/.e->ew; fwx=Series[(fw-fx)/(ew-e),{e,0,5}]//Simplify;
ey=Series[e-fx/fwx,{e,0,5}]//Simplify;fy=fx/.e->ey;
t=Series[fy/fx,{e, 0, 5}]//Simplify;
e1=Series[ey-(1+2t+at^2)fy/fwx,{e,0,5}];
Table[Coefficient[e1,e,i],{i,0,4}]//Simplify

Out[1]={0, 0, 0, 0, -c2((-14+a)c2^2+8c3+4c2f1a^2)}
```

The coefficients in the development of $\widehat{\varepsilon} =$ e1 with $\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3$ are equal to 0, while the nonzero coefficient with ε^4 points to the fourth order of convergence and gives the asymptotic error constant. □

Observe that some of the two-point methods of form (25), that do not satisfy the optimality conditions (24), become optimal with the change of w. For example, method (5) can be written in the form (25) for $h(t,u) = h(t) = 1/(t-1)^2$; it is of order of convergence 3 for $w = x + f(x)$. The method (5) is accelerated to the optimal order 4 taking $w = x + f(x)^2$. Other examples for the weight function of simple form and same convergence behavior are $h(t) = 1 + 2t$, $h(t) = 1/(1-2t)$, $h(t) = (1+t)/(1-t)$, etc. Note

$$h(t) = \frac{1+at}{1+(a-2)t} \quad (a \in \mathbb{R})$$

gives IF of transformed King's method [17] by the transformation $\mathcal{T}$. Methods described in Theorem 3 can be regarded as $\mathcal{T}$-modification of Chun–Petković family of methods, see [12] and [13].

An equivalent statement to Theorem 2 for the three-point methods involves weight functions of two and three variables and more complicated conditions.

Theorem 4. *Let* $(t,u) \mapsto h(t,u)$ *and* $(t,u,v) \mapsto H(t,u,v)$ *be sufficiently differentiable weight functions in the neighborhood of the points* $(0,0)$ *and* $(0,0,0)$, *respectively. If*

$$\begin{aligned}
&h(0,0) = h_t(0,0) = h_v(0,0) = 1,\\
&H(0,0,0) = H_t(0,0,0) = H_u(0,0,0) = H_v(0,0,0) = 1,\\
&H_{tt}(0,0,0) = h_{tt}(0,0), \qquad H_{uu}(0,0,0) = h_{uu}(0,0),\\
&H_{tu}(0,0,0) = 1 + h_{tu}(0,0), \qquad H_{tv}(0,0,0) = H_{uv}(0,0,0) = 2\\
&H_{ttt}(0,0,0) = -6 + 3h_{tt}(0,0) + h_{ttt}(0,0),\\
&H_{uuu}(0,0,0) = -6 + 3h_{uu}(0,0) + h_{uuu}(0,0),\\
&H_{ttu}(0,0,0) = -2 + 2h_{tu}(0,0) + h_{tt}(0,0) + h_{ttu}(0,0),\\
&H_{tuu}(0,0,0) = -2 + h_{uu}(0,0) + 2h_{tu}(0,0) + h_{tuu}(0,0),
\end{aligned}$$

then the three-point family $(\gamma \neq 0)$

$$\begin{cases} w = x + \gamma f(x), \qquad y = x - \dfrac{f(x)}{f[w,x]}, \\ z = y - h\left(\dfrac{f(y)}{f(x)}, \dfrac{f(y)}{f(w)}\right)\dfrac{f(y)}{f[w,x]}, \\ \widehat{x} = z - H\left(\dfrac{f(y)}{f(x)}, \dfrac{f(y)}{f(w)}, \dfrac{f(z)}{f(y)}\right)\dfrac{f(z)}{f[w,x]}, \end{cases} \tag{26}$$

is of optimal order 8.

A new approximation $w = x + f(x)^3$ again relaxes the optimality criteria to a certain extent, by lowering the number of variables and conditions.

Theorem 5. *If* $w = x + f(x)^3$ *and the following conditions are valid*

$$h(0) = 1, \quad h'(0) = 2, \quad H(0,0) = H_v(0,0) = 1, \quad H_t(0,0) = 2,$$
$$H_{tt}(0,0) = 2 + h''(0), \quad H_{tv}(0,0) = 4, \quad H_{ttt}(0,0) = 6h''(0) + h'''(0) - 24,$$

then the three-point family

$$\begin{cases} w = x + f(x)^3, \qquad y = x - \dfrac{f(x)}{f[w,x]}, \\ z = y - h\left(\dfrac{f(y)}{f(x)}\right)\dfrac{f(y)}{f[w,x]}, \\ \widehat{x} = z - H\left(\dfrac{f(y)}{f(x)}, \dfrac{f(z)}{f(y)}\right)\dfrac{f(z)}{f[w,x]}, \end{cases}$$

is of optimal order 8.

Both Theorems 4 and 5 are easily proved with the help of symbolic computation and programming codes in a similar way as it was done in the case of Theorem 3. Note that methods proposed in Theorem 5 can be regarded as $\mathcal{T}$-transformation of the general three-point family presented in [3, eq. (50)].

Remark 2. Taking $w = x + f(x)^m$, $m > 2$ for two-point methods and $m > 3$ for three-point methods, will not further relax optimality conditions of Theorems 3 and 5.

IF (13) in the form (26) is expressed by $h(t,u) = h(t) = 1/(t-1)^2$ and

$$H(t,u,v) = H(t,v) = \frac{1 + t^2(1-v)}{(t-1)^2(tv-1)^2(1-v)}.$$

As seen in Section 3, this IF is accelerated to optimal order 8 when $w = x + f(x)^m$, $m \geq 3$. It can be easily proved that iterative method (13) for $w = x + f(x)$ is of order of convergence 5. The following examples of weight functions in (26) show the same behavior:

(1) $h(t) = 1/(1-2t)$, $H(t,v) = h(t)\Big(1 + \big(v + t^2\big)h(t)\Big)$;

(2) $h(t) = \frac{1+t}{1-t}$, $H(t,v) = h(t)\Big(1 + h(t)\frac{v+t^2+4tv}{1+4t}\Big)$;

(3) $h(t) = 1 + 2t$, $H(t,v) = h(t)\Big(1 + v + h(t)\frac{t^2+2tv}{1+8t}\Big)$;

(4) $h(t) = \frac{1+at}{1+(a-2)t}$, $H(t,v) = h(t)\Big(1 + h(t)\frac{v+t^2+4atv}{1+4at}\Big)$, $a \in \mathbb{R}$;

(5) $h(t) = \frac{1+at}{1+(a-2)t}$,
$H(t,v) = h(t)\frac{1+2(2a-1)t+(1+b)v+(1+c)t^2+(4a+d)tv}{1+2(2a-1)t+bv+ct^2+dtv}$, $a,b,c,d \in \mathbb{R}$.

6. Numerical examples

In numerical implementation initial approximations can be chosen by the following formula given in [18,19] (so-called method of numerical integration)

$$x_0 = \xi + \operatorname{sgn}\{f(a)\}\frac{h}{2}\sum_{j=1}^{N_0-1}\operatorname{sgn}\{f(\xi + (2j - N_0)h/2)\}, \tag{27}$$

where $h = (b-a)/N_0$ and $\xi = (a+b)/2$. The initial approximation becomes closer to the zero of $f(x) = 0$ as the number N_0 increases. In this paper we have used $N_0 = 10$.

For the comparison purpose with $\mathcal{R}_n$, $n = 2,3,4$, several Newton-based methods have been tested along with their $\mathcal{T}$-modifications. Two derivative free families of methods [20] and $\mathcal{P}_n$ were included in the tests, too. We give explicit IF of two- and three-point methods and provide the $\mathcal{T}$-modified IF, where applicable. Four-point methods have been tested to confirm the assertion of Theorem 1 for the case $n = 4$. However, these IF are of quite complex structure and have little practical importance. For this reason we listed those IF in a more condensed form.

Table 1
Two-point methods and their modifications.

Name	Original IF	Modified IF
Chun [12] *CIF*	$\begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ \widehat{x} = y - \dfrac{f(x)+2f(y)}{f(x)}\dfrac{f(y)}{f'(x)} \end{cases}$	$\begin{cases} w = x + f(x)^2,\ y = x - \dfrac{f(x)}{f[w,x]}, \\ \widehat{x} = y - \dfrac{f(x)+2f(y)}{f(x)}\dfrac{f(y)}{f[w,x]} \end{cases}$
Ostrowski [21] *OIF*	$\begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ \widehat{x} = y - \dfrac{f(x)}{f(x)-2f(y)}\dfrac{f(y)}{f'(x)} \end{cases}$	$\begin{cases} w = x + f(x)^2,\ y = x - \dfrac{f(x)}{f[w,x]}, \\ \widehat{x} = y - \dfrac{f(x)}{f(x)-2f(y)}\dfrac{f(y)}{f[w,x]} \end{cases}$

Table 2
Three-point methods and their modifications.

Original IF	Modified IF
Sharma [24] *SIF*	
$\begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ z = y - \dfrac{f(x)}{f(x)-2f(y)}\dfrac{f(y)}{f'(x)}, \\ \widehat{x} = z - \left(1+\dfrac{f(z)}{f(x)}\right)\dfrac{f[x,y]f(z)}{f[y,z]f[x,z]} \end{cases}$	$\begin{cases} w = x + f(x)^3,\ y = x - \dfrac{f(x)}{f[w,x]}, \\ z = y - \dfrac{f(x)}{f(x)-2f(y)}\dfrac{f(y)}{f[w,x]}, \\ \widehat{x} = z - \left(1+\dfrac{f(z)}{f(x)}\right)\dfrac{f[x,y]f(z)}{f[y,z]f[x,z]}. \end{cases}$
Bi–Wu–Ren [25] *BWR*	
$\begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ z = y - \dfrac{2f(x)-f(y)}{2f(x)-5f(y)}\dfrac{f(y)}{f'(x)}, \\ \widehat{x} = z - \dfrac{f(x)+f(z)}{f(x)-f(z)} \\ \quad \times \dfrac{f(z)}{f[y,z]+f[x,x,z](z-y)}. \end{cases}$	$\begin{cases} w = x + f(x)^3,\ y = x - \dfrac{f(x)}{f[w,x]}, \\ z = y - \dfrac{2f(x)-f(y)}{2f(x)-5f(y)}\dfrac{f(y)}{f[w,x]}, \\ \widehat{x} = z - \dfrac{f(x)+f(z)}{f(x)-f(z)} \\ \quad \times \dfrac{f(z)}{f[y,z]+f[w,x,z]\frac{(z-w)(z-y)}{z-x}}. \end{cases}$

First we give the reviews of two-point and three-point methods in Table 1 and Table 2, respectively. Note that two-point methods of Ostrowski [21] and Chun [12] (Table 1) make particular members of King's more general family [17]

$$\begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ \widehat{x} = y - \dfrac{f(x)+\beta f(y)}{f(x)+(\beta-2)f(y)}\dfrac{f(y)}{f'(x)}. \end{cases}$$

Four-point methods and their modification are given below.

- Kung–Traub's four-point method $\mathcal{R}_4$ and its modification

$$\mathcal{R}_4 : \begin{cases} y = \mathcal{N}(x), \\ z = y + \dfrac{\mathcal{F}[f(x), f(y)] + \frac{y-x}{f(x)}}{f(y)-f(x)} f(x)^2, \\ v = z - \dfrac{\mathcal{F}[f(x), f(y), f(z)] - \frac{z-y}{f(x)^2}}{f(z)-f(x)} f(x)^2 f(y), \\ \widetilde{x} = v - \dfrac{\mathcal{F}[f(x), f(y), f(z), f(v)] + \frac{v-z}{f(x)^2 f(y)}}{f(v)-f(x)} f(x)^2 f(y) f(z). \end{cases}$$

$$\mathcal{T}(\mathcal{R}_4) : \begin{cases} w = x + f(x)^4, \qquad y = \mathcal{S}(x), \\ z = y + \dfrac{\mathcal{F}[f(x), f(y)] + \frac{y-x}{f(x)}}{f(y)-f(x)} f(x)^2, \\ v = z - \dfrac{\mathcal{F}[f(x), f(y), f(z)] - \frac{z-y}{f(x)^2}}{f(z)-f(x)} f(x)^2 f(y), \\ \widetilde{x} = v - \dfrac{\mathcal{F}[f(x), f(y), f(z), f(v)] + \frac{v-z}{f(x)^2 f(y)}}{f(v)-f(x)} f(x)^2 f(y) f(z). \end{cases}$$

- Particular four-point member of Neta's family [22] and its derivative free modification

$$NIF: \begin{cases} y = \mathcal{N}(x), \\ z = y - \dfrac{f(x)+2f(y)}{f(x)} \dfrac{f(y)}{f'(x)}, \\ v = z - \dfrac{\mathcal{F}[f(x), f(y), f(z)] - \frac{z-y}{f(x)^2}}{f(z)-f(x)} f(x)^2 f(y), \\ \widetilde{x} = v - \dfrac{\mathcal{F}[f(x), f(y), f(z), f(v)] + \frac{v-z}{f(x)^2 f(y)}}{f(v)-f(x)} f(x)^2 f(y) f(z). \end{cases}$$

$$\mathcal{T}(NIF): \begin{cases} w = x + f(x)^4, \qquad y = \mathcal{S}(x), \\ z = y - \dfrac{f(x)+f(y)}{f(x)-f(y)} \dfrac{f(y)}{f[w,x]}, \\ v = z - \dfrac{\mathcal{F}[f(x), f(y), f(z)] - \frac{z-y}{f(x)^2}}{f(z)-f(x)} f(x)^2 f(y), \\ \widetilde{x} = v - \dfrac{\mathcal{F}[f(x), f(y), f(z), f(v)] + \frac{v-z}{f(x)^2 f(y)}}{f(v)-f(x)} f(x)^2 f(y) f(z). \end{cases}$$

- Particular four-point IF of Petković's general n-point family [15]

$$PIF: \begin{cases} y = \mathcal{N}(x), \\ z = y - \dfrac{f(x)}{f(x)-2f(y)} \dfrac{f(y)}{f'(x)}, \\ v = z - \dfrac{f(z)}{f'(x)\frac{y-z}{x-y} + f[x,y]\frac{x-2y+z}{x-y} - f[x,y,z](x+2y-3z)}, \\ \widetilde{x} = v - f(v)\Big/\Big(f'(x)\dfrac{(v-y)(v-z)}{(y-x)(x-z)} \\ \quad + f[x,y]\dfrac{v^2+x^2+2yz-v(y+z)-x(y+z)}{(x-y)(x-z)} \\ \quad + f[x,y,z]\Big(3v - x - 2y + \dfrac{(v-x)(v-y)}{x-z}\Big) \\ \quad + f[x,y,z,v](4v^2+2yz+x(y+z)-v(2x+3y+3z))\Big). \end{cases}$$

$$\mathcal{T}(PIF): \begin{cases} w = x + f(x)^4, \qquad y = \mathcal{S}(x), \\ z = y - \dfrac{f(x)}{f(x)-2f(y)} \dfrac{f(y)}{f[w,x]}, \\ v = z - \dfrac{f(z)}{f[w,x]\frac{y-z}{x-y} + f[x,y]\frac{x-2y+z}{x-y} - f[x,y,z](x+2y-3z)}, \\ \widetilde{x} = v - f(v)\Big/\Big(f[w,x]\dfrac{(v-y)(v-z)}{(y-x)(x-z)} \\ \quad + f[x,y]\dfrac{v^2+x^2+2yz-v(y+z)-x(y+z)}{(x-y)(x-z)} \\ \quad + f[x,y,z]\Big(3v - x - 2y + \dfrac{(v-x)(v-y)}{x-z}\Big) \\ \quad + f[x,y,z,v](4v^2+2yz+x(y+z)-v(2x+3y+3z))\Big). \end{cases}$$

In Table 3 we used symbol $N_n(t; w, x, \ldots)$ to represent direct Newton's interpolating polynomial of degree n. In each n-point method of order 2^n, $n = 2, 3, 4$, the exponent m of the transformation $w = x + f(x)^m$ was taken to be $m = n$.

We have taken test functions $f_i(x)$ $(i = 1, 2, 3, 4)$ listed below, each of which with a given interval (a, b) that contains a simple root α of f.

$$\begin{array}{ll} f_1(x) = (x-1)\big(x+1+\log(2+x+x^2)\big), & (a,b) = (0,3) \\ f_2(x) = x - (1/3)\exp(-3x+1), & (a,b) = (-1,1) \\ f_3(x) = -20x^5 - \dfrac{x}{2} + \dfrac{1}{2}, & (a,b) = (-1,4) \\ f_4(x) = \exp(\sin(8x)) - 4x, & (a,b) = (-2,4). \end{array}$$

Table 3
Derivative free n-point families.

n	Kung–Traub $\mathcal{P}_n$ [7]	Zheng–Li–Huang ZLH_n [20]
2	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ \widetilde{x} = \mathcal{P}_2(0) \end{cases}$	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ \widetilde{x} = y - \dfrac{f(y)}{N_2'(y; w, x, y)}. \end{cases}$
3	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ z = \mathcal{P}_2(0), \\ \widetilde{x} = \mathcal{P}_3(0). \end{cases}$	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ z = y - \dfrac{f(y)}{N_2'(y; w, x, y)}, \\ \widetilde{x} = z - \dfrac{f(z)}{N_3'(z; w, x, y, z)}. \end{cases}$
4	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ z = \mathcal{P}_2(0), \\ v = \mathcal{P}_3(0), \\ \widetilde{x} = \mathcal{P}_4(0) \end{cases}$	$\begin{cases} w = x + f(x), \ y = \mathcal{S}(x), \\ z = y - \dfrac{f(y)}{N_2'(y; w, x, y)}, \\ v = z - \dfrac{f(z)}{N_3'(z; w, x, y, z)}, \\ \widetilde{x} = v - \dfrac{f(v)}{N_4'(v; w, x, y, z, v)}. \end{cases}$

Table 4
Numerical results for the test function $f_1(x)$, $x_0 = 1.05$, $\alpha = 1$.

Two-point IF			Three-point IF			Four-point IF		
IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3
$\mathcal{R}_2$	1.20(−88)	3.99	$\mathcal{R}_3$	4.63(−689)	7.99	$\mathcal{R}_4$	7.64(−5440)	16.00
$\mathcal{T}(\mathcal{R}_2)$	7.85(−75)	3.99	$\mathcal{T}(\mathcal{R}_3)$	1.85(−644)	7.99	$\mathcal{T}(\mathcal{R}_4)$	3.26(−5114)	16.00
$\mathcal{P}_2$	3.42(−67)	3.99	$\mathcal{P}_3$	2.78(−541)	7.99	$\mathcal{P}_4$	1.58(−4339)	16.00
OIF	7.55(−96)	3.99	*SIF*	1.74(−732)	7.99	*PIF*	1.79(−6114)	16.00
$\mathcal{T}(OIF)$	5.55(−73)	3.99	$\mathcal{T}(SIF)$	3.73(−671)	8.00	$\mathcal{T}(PIF)$	1.95(−5353)	16.00
CIF	3.55(−80)	3.99	*BWR*	1.51(−767)	7.99	*NIF*	1.25(−5222)	16.00
$\mathcal{T}(CIF)$	1.13(−83)	4.00	$\mathcal{T}(BWR)$	8.05(−640)	7.99	$\mathcal{T}(NIF)$	1.99(−4885)	16.00
ZLH_2	4.10(−74)	3.99	ZLH_3	1.05(−614)	7.99	ZLH_4	6.84(−4991)	16.00

Table 5
Numerical results for the test function $f_2(x)$, $x_0 = 0.3$, $\alpha = 1/3$.

Two-point IF			Three-point IF			Four-point IF		
IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3
$\mathcal{R}_2$	6.44(−106)	3.99	$\mathcal{R}_3$	3.47(−872)	8.00	$\mathcal{R}_4$	2.70(−6729)	16.00
$\mathcal{T}(\mathcal{R}_2)$	5.92(−87)	3.99	$\mathcal{T}(\mathcal{R}_3)$	2.08(−766)	7.99	$\mathcal{T}(\mathcal{R}_4)$	1.56(−6171)	16.00
$\mathcal{P}_2$	4.52(−85)	4.00	$\mathcal{P}_3$	2.42(−738)	8.00	$\mathcal{P}_4$	2.20(−5693)	16.00
OIF	4.04(−113)	4.00	*SIF*	2.51(−856)	7.99	*PIF*	3.09(−7229)	16.00
$\mathcal{T}(OIF)$	3.25(−88)	3.99	$\mathcal{T}(SIF)$	2.40(−747)	7.99	$\mathcal{T}(PIF)$	4.94(−6664)	16.00
CIF	5.61(−91)	3.99	*BWR*	1.15(−761)	7.99	*NIF*	1.70(−6344)	16.00
$\mathcal{T}(CIF)$	4.38(−84)	3.99	$\mathcal{T}(BWR)$	5.18(−724)	7.99	$\mathcal{T}(NIF)$	1.71(−5793)	16.00
ZLH_2	3.35(−93)	4.00	ZLH_3	7.75(−712)	8.00	ZLH_4	3.45(−6281)	16.00

Outcomes of the numerical examples are given in Tables 4–6 as the results of the third iteration in the form mantissa (exponent) of $|f(x_3)|$ and the computational order of convergence (COC) defined as (see [23])

$$\mathrm{COC} = \frac{\log|f(x_3)/f(x_2)|}{\log|f(x_2)/f(x_1)|},$$

which well approximates the theoretical order of convergence. Due to slow convergence, in Table 7 we presented results of the fourth iteration $|f(x_4)|$.

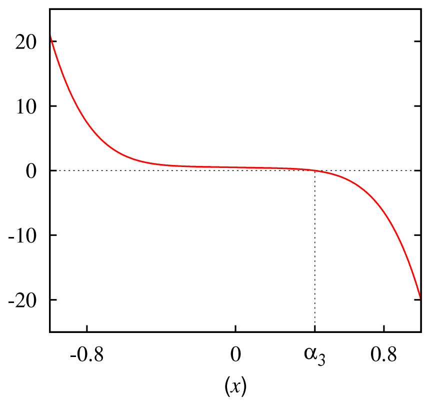
Presented results demonstrate the consistency with the theoretical convergence analysis in Theorem 1. While $f_1(x)$ and $f_2(x)$ are proper test functions for applying any Newton-like iterative method, $f_3(x)$ and $f_4(x)$ behave pathologically near the zeros as shown in Fig. 1. Numerical results given in Table 6 were obtained with considerable efforts in applying the Newton based multipoint methods (very slow convergence or divergence), while the transformed method showed the superiority. For $f_3(x)$ the original method has slower convergence rate than the transformed method. The results for $f_4(x)$ in Table 7 show similar convergence behavior. These results have been obtained taking the initial approximation x_0 not sufficiently close to the zero.

Table 6
Numerical results for the test function $f_3(x)$, $x_0 = 0.25$, $\alpha = 0.4276772969\ldots$.

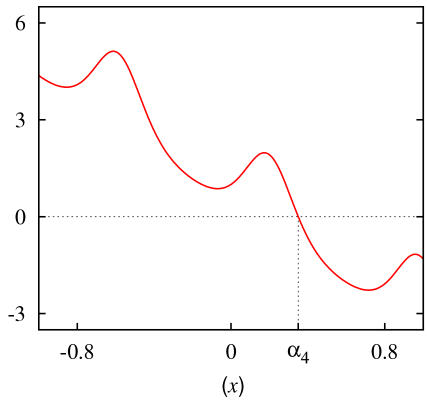
Two-point IF			Three-point IF			Four-point IF		
IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3	IF	$\lvert f(x_3)\rvert$	COC_3
$\mathcal{R}_2$	4.59(−5)	2.66	$\mathcal{R}_3$	1.12(−23)	7.05	$\mathcal{R}_4$	2.30(−138)	15.62
$\mathcal{T}(\mathcal{R}_2)$	1.09(−29)	3.99	$\mathcal{T}(\mathcal{R}_3)$	2.11(−46)	7.90	$\mathcal{T}(\mathcal{R}_4)$	4.01(−196)	15.86
$\mathcal{P}_2$	div.		$\mathcal{P}_3$	1.01(−25)	8.05	$\mathcal{P}_4$	7.03(−278)	16.00
OIF	5.57(−16)	3.89	*SIF*	4.04(−81)	7.97	*PIF*	2.97(−818)	15.99
$\mathcal{T}(OIF)$	2.48(−18)	3.99	$\mathcal{T}(SIF)$	2.66(−175)	8.00	$\mathcal{T}(PIF)$	1.19(−924)	15.99
CIF	div.		*BWR*	div.		*NIF*	4.40(−82)	14.83
$\mathcal{T}(CIF)$	8.92(−1)	3.25	$\mathcal{T}(BWR)$	1.94(−1)	0.25	$\mathcal{T}(NIF)$	4.29(−79)	15.53
ZLH_2	5.88(−15)	3.99	ZLH_3	1.05(−120)	7.99	ZLH_4	1.92(−1013)	15.99

Table 7
Numerical results for the test function $f_4(x)$, $x_0 = 0.1$, $\alpha = 0.3498572166\ldots$.

Two-point IF			Three-point IF			Four-point IF		
IF	$\lvert f(x_4)\rvert$	COC_4	IF	$\lvert f(x_4)\rvert$	COC_4	IF	$\lvert f(x_4)\rvert$	COC_4
$\mathcal{R}_2$	9.63(−1)	2.44	$\mathcal{R}_3$	1.50(0)	0.14	$\mathcal{R}_4$	div.	
$\mathcal{T}(\mathcal{R}_2)$	4.13(−28)	3.92	$\mathcal{T}(\mathcal{R}_3)$	1.16(−665)	7.99	$\mathcal{T}(\mathcal{R}_4)$	6.92(−8490)	16.00
$\mathcal{P}_2$	7.42(−1)	1.80	$\mathcal{P}_3$	1.64(0)	0.74	$\mathcal{P}_4$	6.26(−44)	13.88
OIF	8.80(−2)	2.11	*SIF*	div.		*PIF*	1.06(0)	1.16
$\mathcal{T}(OIF)$	8.81(−7)	2.11	$\mathcal{T}(SIF)$	6.43(−801)	7.99	$\mathcal{T}(PIF)$	9.74(−703)	15.94
CIF	5.66(−6)	0.78	*BWR*	div.		*NIF*	3.76(−12)	9.11
$\mathcal{T}(CIF)$	5.72(−6)	0.78	$\mathcal{T}(BWR)$	9.08(−546)	8.00	$\mathcal{T}(NIF)$	9.58(−22)	7.16
ZLH_2	3.52(−29)	3.99	ZLH_3	7.97(−51)	8.01	ZLH_4	2.30(−1417)	15.99



(a) $f_3(x)$.



(b) $f_4(x)$.

Fig. 1. Graphs of the test functions $f_3(x)$ and $f_4(x)$ having pathological behavior near the zeros $\alpha_3 \approx 0.4276772969$ and $\alpha_4 \approx 0.3498572166$, respectively.

Although the transformed methods studied in this paper were not the best in all tested numerical experiments, most of examples show that the transformed method without derivative can be certainly used to overcome the limit of the existing Newton-like methods for particular cases such as $f_3(x)$ and $f_4(x)$.

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