Journal of Computational and Applied Mathematics 290 (2015) 298–309

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On an efficient inclusion method for finding polynomial zeros☆

Dušan M. Milošević [a], Mimica R. Milošević [b], Jovana Džunić [a,*]

[a] Faculty of Electronic Engineering, University of Niš, 18 000 Niš, Serbia

[b] Faculty of Management in Civil Engineering, University Union Nikola Tesla, 11000 Belgrade, Serbia

ARTICLE INFO

Article history:
Received 26 October 2014
Received in revised form 8 May 2015

MSC:
65H05
65G20
30C15

Keywords:
Polynomial zeros
Simultaneous methods
Inclusion methods
Convergence conditions
Circular interval arithmetic

ABSTRACT

New efficient iterative method of Halley's type for the simultaneous inclusion of all simple complex zeros of a polynomial is proposed. The presented convergence analysis, which uses the concept of the R-order of convergence of mutually dependent sequences, shows that the convergence rate of the basic fourth order method is increased from 4 to 9 using a two-point correction. The proposed inclusion method possesses high computational efficiency since the increase of convergence is attained with only one additional function evaluation per sought zero. Further acceleration of the proposed method is carried out using the Gauss–Seidel procedure. Some computational aspects and three numerical examples are given in order to demonstrate high computational efficiency and the convergence properties of the proposed methods.

1. Introduction

There are only few situations when it is possible to find an analytic solution of nonlinear equations solving real life problems as well as tasks of theoretical nature. The override is found in an algorithmic approach of iterative type constructed to provide an admissible approximate solution within a finite number of rational operations. In this way the point of solution is no longer a zero-dimensional subject, it is given its 'width'—everything within a range of an acceptable error. Iterative methods designed in interval arithmetic make verbatim representation of algorithms producing 'massive' approximating points, assessing deviation from the exact solution at the same time.

The aim of this paper is to explore efficient Halley-like methods for the simultaneous inclusion of all simple complex zeros of a polynomial. It can be regarded as an extension of interval version to Halley-like iterative methods, presented in [1,2], and discussed later in [3–7].

The presented convergence analysis shows that the convergence rate of the basic fourth order method is increased from 4 to 9. The convergence and efficiency outbreak are the aftermath of special type of corrections obtained from a two-point iterative method of low computational complexity. The suggested algorithm achieves remarkable convergence rate with only one additional function calculation per sought zero loop-wise. The presented data usage significantly increases computational efficiency of an iterative method.

☆ This research was supported by the Serbian Ministry of Education and Science under grant number 174022.

* Corresponding author.
E-mail address: jovana.dzunic@gmail.com (J. Džunić).

http://dx.doi.org/10.1016/j.cam.2015.05.015
0377-0427/

Table 1
The number of basic operations.

	(1)	(2)	(3)	(4)
AS	3	6	2	2
M	0	11	3	3
D	0	0	3	3
R	0	2	0	1

Multipoint methods are among the most efficient tools in approximating a single root of a nonlinear equation by iterative means. For details and relevant references the reader is referred to [8,9]. In the present paper we use this powerful tool fit to the given task. Based on the new two step scheme provided in the manuscript, we double the order of convergence of the Halley's single root finding method, while using only one additional function evaluation per iteration. This new two-step scheme is used as a correction in Halley's like interval procedure. This better correction implemented in simultaneous procedure allows us to combine more polynomial information for better results, while economizing on expensive interval arithmetic.

The paper is organized as follows. The basic properties of circular complex arithmetic, necessary for the convergence analysis and the construction of inclusion methods, are given in the Introduction. The basic Halley-like inclusion method of the fourth order, is presented in short in Section 2 and the modified method with the increased convergence rate is developed in Section 3 using a suitable two-point correction. The convergence analysis of the improved method is provided in Section 4. The corresponding single-step methods are prospected in Section 5, while the results overview of computational aspects and several numerical examples are given in Section 6.

For the reader's convenience we give some basic properties of circular complex arithmetic introduced by Gargantini and Henrici [10]. A circular closed region (disk) $\boldsymbol{Z} := \{z : |z - c| \le r\}$ with center $c := \operatorname{mid} \boldsymbol{Z}$ and radius $r := \operatorname{rad} \boldsymbol{Z}$ we will denote by parametric notation $\boldsymbol{Z} := \{c; r\}$. Complex points z are then treated as 'degenerated' discs of center z and radius 0. The set of complex circular intervals (disks) is denoted by $K(\mathbb{C})$.

The basic circular arithmetic operations are defined as follows:

$$\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\}, \tag{1}$$

$$\{c_1; r_1\} \cdot \{c_2; r_2\} = \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\}. \tag{2}$$

The inversion of a non-zero disk $\boldsymbol{Z}$ is defined by the Möbius transformation,

$$\boldsymbol{Z}^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r,\ \text{i.e. } 0 \notin \boldsymbol{Z}). \tag{3}$$

Beside the exact inversion $\boldsymbol{Z}^{-1}$ of a disk $\boldsymbol{Z}$, the so-called *centered inversion* $\boldsymbol{Z}^{I_c}$ defined by

$$\boldsymbol{Z}^{I_c} = \{c; r\}^{I_c} := \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \supseteq \boldsymbol{Z}^{-1} \quad (0 \notin \boldsymbol{Z}) \tag{4}$$

is often used.

Computational costs of operations (1)–(4) in number of additions + subtractions (AS), multiplications (M), divisions (D) and extractions of a root (R) in real arithmetic are given in Table 1.

Using (3) and (4) the discs division is defined as

$$\boldsymbol{Z}_1 : \boldsymbol{Z}_2 = \boldsymbol{Z}_1 \cdot \boldsymbol{Z}_2^{-1} \quad \text{or} \quad \boldsymbol{Z}_1 : \boldsymbol{Z}_2 = \boldsymbol{Z}_1 \cdot \boldsymbol{Z}_2^{I_c} \quad (0 \notin \boldsymbol{Z}_2).$$

If F is a circular complex function and the implication

$$\boldsymbol{Z}_1 \subseteq \boldsymbol{Z}_2 \Longrightarrow F(\boldsymbol{Z}_1) \subseteq F(\boldsymbol{Z}_2)$$

holds, then F is an *inclusion isotone* function. Consequently

$$z \in \boldsymbol{Z} \Longrightarrow F(z) \in F(\boldsymbol{Z}),$$

holds.

More details about circular arithmetic can be found in the books [11,4,12]. Throughout this paper disks in the complex plane will be denoted by bolded capital letters. Vectors of disks will also be denoted by bolded capital letters without risk of confusion.

To estimate the convergence rate of interval inclusion methods for solving equations, we follow the approach proposed by Alefeld and Herzberger [11, Appendix A] and M. Petković [4, Chapter 1] based on the concept of R-order of convergence, introduced by Ortega and Rheinboldt [13]. In this way the lacks of classical definition of order of convergence is overcome.

Assume that $\boldsymbol{Z}^* \in K(\mathbb{C})$ is the limit of the sequence $\{\boldsymbol{Z}^{(m)}\}$ of circular complex intervals belonging to $K(\mathbb{C})$ such that $\boldsymbol{Z}^* \subseteq \boldsymbol{Z}^{(m)}$. One of the possibility to measure the deviation of an element $\boldsymbol{Z}^{(m)}$ of the sequence produced by an iterative inclusion method $\Im$ from the limit $\boldsymbol{Z}^*$ can be expressed by a nonnegative real number

$$h^{(m)} = d(\boldsymbol{Z}^{(m)}) - d(\boldsymbol{Z}^*),$$

where $d(\boldsymbol{Z}^{(m)}) := 2 \operatorname{rad} \boldsymbol{Z}^{(m)}$ (see [11, Appendix A]). In particular, if $\boldsymbol{Z}^* = z^*$ is a point in the complex plane, we have

$$h^{(m)} = d(\boldsymbol{Z}^{(m)}) = 2 \operatorname{rad} \boldsymbol{Z}^{(m)}.$$

We note that $\{h^{(m)}\}$ is the *null* sequence.

Definition 1.1. Let $\{\boldsymbol{Z}^{(m)}\}$ be the sequence of disks generated by the simultaneous inclusion method $\Im$ that converges to z^*. The R-factor $R_s\{\boldsymbol{Z}^{(m)}\}$ of the sequence for the real number $s \ge 1$ is defined by

$$R_s\{\boldsymbol{Z}^{(m)}\} = \begin{cases} \lim\limits_{m\to\infty} \sup\big(h^{(m)}\big)^{1/m}, & s = 1, \\ \lim\limits_{m\to\infty} \sup\big(h^{(m)}\big)^{1/s^m}, & s > 1. \end{cases}$$

Definition 1.2. Let $\mathcal{C}(\Im, z^*)$ be the set of all sequences produced by the method $\Im$ for which

$$\lim_{m\to\infty} \boldsymbol{Z}^{(m)} = z^* \quad \text{and} \quad z^* \in \boldsymbol{Z}^{(m)} \quad (m = 0, 1, \ldots).$$

The R-factor $R_s(\Im, z^*)$ of $\Im$ at z^* is defined by

$$R_s(\Im, z^*) = \sup\Big\{R_s\big(\boldsymbol{Z}^{(m)}\big) : \big\{\boldsymbol{Z}^{(m)}\big\} \in \mathcal{C}(\Im, z^*)\Big\}.$$

Definition 1.3. The R-order $O_R(\Im, z^*)$ of $\Im$ at z^* is defined by

$$O_R(\Im, z^*) = \begin{cases} +\infty, & \text{if } R_s(\Im, z^*) = 0, \text{ for all } s \ge 1, \\ \inf\{s : s \in [1, \infty),\ R_s(\Im, z^*) = 1\}, & \text{otherwise.} \end{cases}$$

2. Total-step Halley-like method

Let us consider a monic polynomial of degree $n \ge 3$

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n}(z - \zeta_j), \quad (a_i \in \mathbb{C})$$

with simple zeros $\zeta_1, \ldots, \zeta_n$ and let

$$\sigma_{k,i} := \sum_{\substack{j=1 \\ j\ne i}}^{n} (z_i - \zeta_j)^{-k} \quad (k = 1, 2,\ i \in \mathbb{I}_n := \{1, \ldots, n\}).$$

The vector of zeros will be denoted by $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$.

Using the concept of Bell's polynomials, Wang and Zheng derived in [2] the following fixed-point relation

$$\zeta_i = z_i - \frac{1}{H(z_i)^{-1} - \frac{P(z_i)}{2P'(z_i)}\Big(\sigma_{1,i}^2 + \sigma_{2,i}\Big)} \quad (i \in \mathbb{I}_n), \tag{5}$$

where

$$H(z_i) = \left(\frac{P'(z_i)}{P(z_i)} - \frac{P''(z_i)}{2P'(z_i)}\right)^{-1}.$$

Let us define the disks

$$\boldsymbol{S}_{k,i}(\boldsymbol{X}, \boldsymbol{W}) := \sum_{j=1}^{i-1}\Big(\mathrm{INV}_1(z_i - \boldsymbol{X}_j)\Big)^k + \sum_{j=i+1}^{n}\Big(\mathrm{INV}_1(z_i - \boldsymbol{W}_j)\Big)^k \quad (k = 1, 2), \tag{6}$$

where $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ and $\boldsymbol{W} = (\boldsymbol{W}_1, \ldots, \boldsymbol{W}_n)$ are vectors which components are disks and $\mathrm{INV}_1 \in \{()^{-1}, ()^{I_c}\}$.

Suppose that n disjoint disks $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ such that $\zeta_j \in \boldsymbol{Z}_j$ $(j \in \mathbb{I}_n)$ have been found. Let us put $z_i = \mathrm{mid}\, \boldsymbol{Z}_i$ in (5). Since $\zeta_j \in \boldsymbol{Z}_j$ $(j \in \mathbb{I}_n)$, according to the inclusion isotonicity property we obtain

$$\zeta_i \in z_i - \frac{1}{H(z_i)^{-1} - \frac{P(z_i)}{2P'(z_i)}\Big(\boldsymbol{S}_{1,i}^2(\boldsymbol{Z}, \boldsymbol{Z}) + \boldsymbol{S}_{2,i}(\boldsymbol{Z}, \boldsymbol{Z})\Big)} \quad (i \in \mathbb{I}_n). \tag{7}$$

Let $\boldsymbol{Z}_1^{(0)}, \ldots, \boldsymbol{Z}_n^{(0)}$ be initial disjoint disks containing the zeros $\zeta_1, \ldots, \zeta_n$, that is, $\zeta_i \in \boldsymbol{Z}_i^{(0)}$ for all $i \in \mathbb{I}_n$. The relation (7) suggests the following methods for the simultaneous inclusion of all zeros of P:

$$\boldsymbol{Z}_i^{(m+1)} = z_i^{(m)} - \mathrm{INV}_2\bigg(\Big[H(z_i^{(m)})\Big]^{-1} - \frac{P(z_i^{(m)})}{2P'(z_i^{(m)})}\Big[\boldsymbol{S}_{1,i}^2(\boldsymbol{Z}^{(m)}, \boldsymbol{Z}^{(m)}) + \boldsymbol{S}_{2,i}(\boldsymbol{Z}^{(m)}, \boldsymbol{Z}^{(m)})\Big]\bigg), \tag{8}$$

where $m = 0, 1, \ldots, i \in \mathbb{I}_n$ and $\mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$. The subscript indices "1" and "2" point to the order of application of the inversion; namely, in the realization of the iterative formula (8) we first apply the inversion INV_1 to the sums (6), and then the inversion INV_2 in the final step.

Remark 2.1. The main part in the iterative formula (8) is Halley's correction $H(z)$. For this reason, this method, as well as its modification which will be considered in this paper, are referred to as *Halley-like* methods.

The iterative method (8) was extensively studied in [1,2], where it was proved that the order of convergence of the method (8) is *four*.

3. The improved inclusion method

The interval inclusion method (8) can be accelerated following Nourein's approach with corrections [14], explored in papers [15,16]

$$\boldsymbol{Z}_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left[H(z_i^{(m)})\right]^{-1} - \frac{1}{2}N(z_i^{(m)})\left[\boldsymbol{S}_{1,i}^2(\boldsymbol{Z}_C^{(m)}, \boldsymbol{Z}_C^{(m)}) + \boldsymbol{S}_{2,i}(\boldsymbol{Z}_C^{(m)}, \boldsymbol{Z}_C^{(m)})\right]}, \tag{9}$$

($i \in \mathbb{I}_n$), where

$$\boldsymbol{Z}_C^{(m)} = \left(\boldsymbol{Z}_{C,1}^{(m)}, \dots, \boldsymbol{Z}_{C,n}^{(m)}\right), \qquad \boldsymbol{Z}_{C,i}^{(m)} = \boldsymbol{Z}_i^{(m)} - C\left(z_i^{(m)}\right),$$

and $C(z)$ is Newton's

$$N(z) = \frac{P(z)}{P'(z)}$$

or Halley's correction

$$H(z) = \left(\frac{P'(z)}{P(z)} - \frac{P''(z)}{2P'(z)}\right)^{-1},$$

which appear in the well-known iterative formulas $z^{(m+1)} = z^{(m)} - N\left(z^{(m)}\right)$ and $z^{(m+1)} = z^{(m)} - H\left(z^{(m)}\right)$ with the convergence order two and three, respectively.

When the exact inversions ($\text{INV}_1 = \text{INV}_2 = ()^{-1}$) are used in (9), it was proved in [5] that the lower bound of the R-order of convergence of the interval method (9) is $O_R(9) \geq 2 + \sqrt{7} \cong 4.646$. The analysis presented in [5] (see, also, [1]) shows that further increase of convergence speed for the method (9) is quite limited when the exact inversion INV_1 is applied to the sums (6).

In the case when the centered inversions ($\text{INV}_1 = \text{INV}_2 = ()^{I_c}$) are used in (9), it was proved in [6] that the lower bound of the R-order of convergence of the interval method (9) is $O_R(9) \geq 5$ or 6 depending on the used correction (Newton's or Halley's), respectively.

Further improvement in convergence and efficiency rate can be obtained using even higher order correction $C(z)$. Multipoint methods play the crucial role in the correction design. Their aim is to bring out the most from the available information. With such an approach we combine two powerful tools—simultaneous approximation and multipoint methods, to exhaust information for the best results.

Remark 3.1. In the sequel we will concentrate only on the centered inversion (4) used in (9). The reason for such a choice was minutely analyzed and explained in [5]. In short, the exact inversion (3) does give smaller disks. However, the application of the centered inversion produces centers of resulting disks that are a much better approximating solution. In return, this central convergence forces the contraction of the disks which leads to the accelerated convergence of interval methods. Shifted centers obtained in the central inversion oppose convergence increase when the higher order corrections are implemented.

In this paper we consider a two-point correction based on Halley's iteration

$$\begin{cases} y = z - H(z), \\ C(z) = H(z) + \dfrac{P(y)}{h_3'(y)}, \end{cases} \tag{10}$$

where

$$\begin{aligned} h_3'(y) &= 3\,\frac{P(y) - P(z)}{y - z} - 2P'(z) - \frac{P''(z)}{2}(y - z) \\ &= P'(z)\left[3H(z)^{-1}(N(z) - V(z)) + N(z)^{-1}H(z) - 3\right]. \end{aligned} \tag{11}$$

Here h_3 is Hermite's interpolating polynomial of third degree satisfying the conditions

$$h_3(z) = P(z), \qquad h_3'(z) = P'(z), \qquad h_3''(z) = P''(z), \qquad h_3(y) = P(y),$$

Table 2
The number of basic operations.

	(8)	$(9)_N$	$(9)_H$	$(9)_{TPC}$
AS(n)	$23n^2$	$25n^2-2n$	$25n^2-2n$	$29n^2+3n$
M(n)	$32n^2-n$	$32n^2-n$	$32n^2-n$	$38n^2$
D(n)	$6n^2+n$	$6n^2+n$	$6n^2+n$	$6n^2+3n$
R(n)	$2n^2+n$	$2n^2+n$	$2n^2+n$	$2n^2+n$

where $N(z)=\frac{P(z)}{P'(z)}$, $V(z)=\frac{P(y)}{P'(z)}$. Substituting (11) in (10) we obtain the sixth-order correction in the form

$$\begin{cases} y = z - H(z), \qquad V(z) = \dfrac{P(y)}{P'(z)}, \\ C(z) = H(z) + \dfrac{V(z)}{3H(z)^{-1}(N(z)-V(z)) + N(z)^{-1}H(z) - 3}, \end{cases} \tag{12}$$

which combines already computed expressions $N(z)$ and $H(z)$ depending on $P(z)$, $P'(z)$, $P''(z)$, and one new polynomial evaluation $P(y)$ involved in $V(z)$.

Remark 3.2. To decrease the total computational cost, before executing any iteration step it is necessary to calculate first all corrections $C\big(z_j^{(m)}\big)$.

Remark 3.3. Evaluations of the polynomial P and its derivatives P' and P'', as well as the necessary expressions such as N, H, V, etc., are calculated in floating-point arithmetic. More precisely, since we deal with methods of very high order we use multiprecision arithmetic which provides considerably high precision of intermediate results. In this way the rounding errors in scalar quantities are avoided and the inclusion property is preserved (each disk contains one zero in each iteration). Even when floating-point arithmetic of lower precision is employed (but to the extend when the radii of disks do not exceed a machine precision), the analyzes of dynamical stability of existing circular interval methods shows that interval methods preserve their order of convergence when the upper bound of rounding errors δ is of the same order as r, where r is the maximal radius of inclusion disks, see the Refs. [17,18,1], [4, pp. 96–106], [19]. Moreover, possible calculation with circular disks, for example $\{H(z_i);\delta_{H,i}\}$ and $\{N(z_i);\delta_{N,i}\}$ instead of $H(z_i)$ and $N(z_i)$ respectively, would make unnecessary complications since we have no information on the value of the "radii" (errors) $\delta_{H,i}$ and $\delta_{N,i}$. On the other hand, inclusion disks are naturally defined by their radii. Fortunately, we need not artificial radii/errors, which is confirmed in practice in many papers on the topic. For the above-mentioned reasons, in this paper we proceed as the authors of existing papers did (see the book [4] and references cited there) and calculate scalar quantities in floating-point methods of sufficiently high precision. Such approach gives more efficient methods, preserves order of convergence and inclusion property; this facts are confirmed in practice. Consequently, Table 2 of basic operations deals with scalar quantities P, P', P'', N, H, V, etc., not with corresponding circular disks.

4. Convergence analysis of the improved method

In this section we give the convergence analysis of the interval method (9) with the two-point correction (12). For simplicity, we omit the iteration index m and denote all quantities at the $(m+1)$-st iteration with symbol $\widehat{\ }$. To estimate the order of convergence of the iterative method we introduce the errors

$$\varepsilon_i = z_i - \zeta_i \quad (i \in \mathbb{I}_n).$$

It is assumed that $\varepsilon_i = \mathcal{O}_M(\varepsilon_j)$ for any pair $i,j\in\mathbb{I}_n$. The symbol $\mathcal{O}_M$ points to the fact that two real or complex numbers ω_1 and ω_2 have moduli of the same order (that is, $|\omega_1| = \mathcal{O}(|\omega_2|)$). For brevity, we will write $\sum_{j\neq i}$ instead of $\sum_{\substack{j=1\\ j\neq i}}^{n}$.

We require assertions of the following two lemmas.

Lemma 4.1. *Let $f(x)$ be a sufficiently differentiable function in a neighborhood of a simple zero α of f. Iteration function $\widehat{x}=\Phi(x)$ defined with*

$$\begin{cases} y = x - H(x), \\ \widehat{x} = y - \dfrac{f(y)}{h_3'(y)}, \end{cases} \tag{13}$$

is of order six, where $h_3(t)$ is Hermite's interpolating polynomial of third degree satisfying the conditions

$$h_3(x) = f(x), \qquad h_3'(x) = f'(x), \qquad h_3''(x) = f''(x), \qquad h_3(y) = f(y).$$

Proof. Let us introduce the following notation

$$e = x - \alpha, \qquad e_y = y - \alpha, \qquad \widehat{e} = \widehat{x} - \alpha.$$

Since the first step of (13) represents a third order Halley's iteration, we have $e_y = \mathcal{O}_M\big(e^3\big)$. Based on Cauchy's theorem, a remainder relation for Hermite's interpolation states

$$f(t) - h_3(t) = \frac{f^{(4)}(\xi)}{4!}(t-x)^3(t-y), \quad \xi \in I(t,x,y), \tag{14}$$

where $I(t,x,y)$ denotes the smallest interval containing points t, x, y. When t, x and y are chosen close to α this implies for ξ as well. After differentiating (14) and taking $t = y$, we obtain

$$f'(y) - h_3'(y) = \frac{f^{(4)}(\xi)}{4!}(y-x)^3 = \frac{f^{(4)}(\alpha) + \mathcal{O}_M(\xi - \alpha)}{4!}(e_y - e)^3 = \mathcal{O}_M(e^3)$$

because of $(e_y - e)^3 = (\mathcal{O}_M(e^3) - e)^3 = \mathcal{O}_M(e^3)$. Since α is a simple zero of f, there exists a neighborhood of α such that $f'(y) \neq 0$, therefore $f'(y) = \mathcal{O}_M(1)$, thus we have

$$h_3'(y) = f'(y)\big(1 + \mathcal{O}_M(e^3)\big). \tag{15}$$

By means of Taylor's expansion of the function f in the neighborhood of the sought zero α and using (15) it follows

$$\widehat{x} = y - \frac{f(y)}{h_3'(y)} = y - \frac{f(y)}{f'(y)}\big(1 + \mathcal{O}_M(e^3)\big) = y - N(y) + \mathcal{O}_M(f(y)e^3).$$

Since $f(y) = f'(\alpha)e_y + \frac{f''(\alpha)}{2}e_y^2 + \cdots = \mathcal{O}_M(e_y) = \mathcal{O}_M(e^3)$, and from the well known error relation of Newton's method $y - N(y) - \alpha = \mathcal{O}_M(e_y^2)$, we have

$$\widehat{e} = \mathcal{O}_M(e_y^2) + \mathcal{O}_M\big(e_y e^3\big) = \mathcal{O}_M\big(e^6\big). \quad \square$$

In the sequel we will use the following abbreviations:

$$\varepsilon = \max_{1\le i\le n} |\varepsilon_i|, \qquad r = \max_{1\le i\le n} r_i, \qquad \delta_{k,i} = \sum_{j=1}^{n} \frac{1}{(z_i - \zeta_j)^k} \quad (k = 1, 2).$$

Then the inclusion method (9) takes the form

$$\boldsymbol{Z}_i^{(m+1)} = z_i^{(m)} - \frac{2\delta_{1,i}^{(m)}}{\big(\delta_{1,i}^{(m)}\big)^2 - \boldsymbol{S}_{1,i}^2(\boldsymbol{Z}_C^{(m)}, \boldsymbol{Z}_C^{(m)}) + \delta_{2,i}^{(m)} - \boldsymbol{S}_{2,i}(\boldsymbol{Z}_C^{(m)}, \boldsymbol{Z}_C^{(m)})} \quad (i \in \mathbb{I}_n), \tag{16}$$

where

$$N(\zeta_i)^{-1} = \delta_{1,i}, \qquad H(\zeta_i)^{-1} = \frac{1}{2}(\delta_{1,i}^2 + \delta_{2,i})/\delta_{1,i}.$$

Lemma 4.2. *Let $r = o\big(\min_{i,j} |\zeta_i - \zeta_j|\big)$. Then for the inclusion method (16), with the two-point correction (10), the following relations are true:*

(i) $\widehat{r} = \mathcal{O}_M\big(\varepsilon^3 r\big)$;
(ii) $\widehat{\varepsilon} = \mathcal{O}_M\big(\varepsilon^9\big)$.

Proof. Let $\boldsymbol{Z}_j = \{z_j; r_j\}$, $C_j = C(z_j)$ and $z_i - \boldsymbol{Z}_j + C_j = \{\eta_{ij}; r_j\}$, where $\eta_{ij} = z_i - \Phi(z_j)$ and $\Phi(z_j)$ is defined in (13) of Lemma 4.1.

First, let us examine the difference

$$\begin{aligned}\sigma_{1,i} - \boldsymbol{S}_{1,i} &= \sum_{j\neq i} \frac{1}{z_i - \zeta_j} - \sum_{j\neq i} \frac{1}{\{\eta_{ij}; r_j\}} \\ &= \sum_{j\neq i} \frac{1}{z_i - \zeta_j} - \sum_{j\neq i} \left\{\frac{1}{\eta_{ij}}; \frac{r_j}{|\eta_{ij}|(|\eta_{ij}| - r_j)}\right\} \\ &= \sum_{j\neq i} \left\{\frac{\zeta_j - \Phi(z_j)}{(z_i - \zeta_j)\eta_{ij}}; \frac{r_j}{|\eta_{ij}|(|\eta_{ij}| - r_j)}\right\} = \{u_i; \rho_i\}.\end{aligned}$$

Since $\zeta_j - \Phi(z_j) = \mathcal{O}_M(\varepsilon_j^6)$ (see Lemma 4.1), $\eta_{ij} = \mathcal{O}_M(1)$ and $z_i - \zeta_j = \mathcal{O}_M(1)$ so that we obtain the following approximations

$$u_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\zeta_j - \Phi(z_j)}{(z_i - \zeta_j)\,\eta_{ij}} = \mathcal{O}_M(\varepsilon^6), \tag{17}$$

and

$$\rho_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{r_j}{|\eta_{ij}|(|\eta_{ij}|-r_j)} = \mathcal{O}_M(r). \tag{18}$$

The difference $\delta_{1,i}^2 - \boldsymbol{S}_{1,i}^2$ is now given by

$$\delta_{1,i}^2 - \boldsymbol{S}_{1,i}^2 = \left(\frac{1}{\varepsilon_i} + \sigma_{1,i}\right)^2 - \boldsymbol{S}_{1,i}^2 = \frac{1}{\varepsilon_i^2} + 2\,\frac{1}{\varepsilon_i}\,\sigma_{1,i} + (\sigma_{1,i} - \boldsymbol{S}_{1,i})(\sigma_{1,i} + \boldsymbol{S}_{1,i}).$$

Using (17) and (18) and having in mind that $\sigma_{1,i} + \boldsymbol{S}_{1,i} = \{\mathcal{O}_M(1); \mathcal{O}_M(r)\}$, we obtain the approximation

$$\delta_{1,i}^2 - \boldsymbol{S}_{1,i}^2 = \left\{\frac{1}{\varepsilon_i^2} + 2\,\frac{1}{\varepsilon_i}\,\sigma_{1,i} + \mathcal{O}_M(\varepsilon^6);\ \mathcal{O}_M(r)\right\}. \tag{19}$$

Similarly, the difference $\boldsymbol{A}_i = \delta_{2,i} - \boldsymbol{S}_{2,i}$ can be expressed in the form

$$\begin{aligned}
\boldsymbol{A}_i &= \frac{1}{\varepsilon_i^2} + \sum_{j\neq i} \frac{1}{(z_i-\zeta_j)^2} - \sum_{j\neq i}\left(\frac{1}{\{\eta_{ij}; r_j\}}\right)^2 \\
&= \frac{1}{\varepsilon_i^2} + \sum_{j\neq i} \frac{1}{(z_i-\zeta_j)^2} - \sum_{j\neq i}\left\{\frac{1}{\eta_{ij}};\ \frac{r_j}{|\eta_{ij}|(|\eta_{ij}|-r_j)}\right\}^2 \\
&= \frac{1}{\varepsilon_i^2} + \sum_{j\neq i} \frac{1}{(z_i-\zeta_j)^2} - \sum_{j\neq i}\left\{\frac{1}{\eta_{ij}^2};\ \frac{2r_j}{|\eta_{ij}|^2(|\eta_{ij}|-r_j)} + \frac{r_j^2}{|\eta_{ij}|^2(|\eta_{ij}|-r_j)^2}\right\} \\
&= \frac{1}{\varepsilon_i^2} + \sum_{j\neq i}\left\{\frac{\zeta_j - \Phi(z_j)}{(z_i-\zeta_j)\eta_{ij}}\left(\frac{1}{z_i-\zeta_j} + \frac{1}{\eta_{ij}}\right);\ \frac{2r_j}{|\eta_{ij}|^2(|\eta_{ij}|-r_j)} + \frac{r_j^2}{|\eta_{ij}|^2(|\eta_{ij}|-r_j)^2}\right\}.
\end{aligned}$$

Since

$$\frac{1}{z_i-\zeta_j} + \frac{1}{\eta_{ij}} = \mathcal{O}_M(1),$$

we have, from (18) and (19),

$$\delta_{2,i} - \boldsymbol{S}_{2,i} = \left\{\frac{1}{\varepsilon_i^2} + \mathcal{O}_M(\varepsilon^6);\ \mathcal{O}_M(r)\right\}. \tag{20}$$

Using (19) and (20), from (16), we find

$$\{\widehat{\varepsilon}_i; \widehat{r}_i\} = \varepsilon_i - \frac{2\left(\frac{1}{\varepsilon_i} + \sigma_{1,i}\right)}{\left\{2\,\frac{1}{\varepsilon_i^2} + 2\,\frac{1}{\varepsilon_i}\sigma_{1,i} + \mathcal{O}_M(\varepsilon^6);\ \mathcal{O}_M(r)\right\}}. \tag{21}$$

According to (21) we obtain

$$\widehat{\varepsilon}_i = \varepsilon_i - \frac{2(\varepsilon_i + \varepsilon_i^2\sigma_{1,i})}{2 + 2\varepsilon_i\sigma_{1,i} + \mathcal{O}_M(\varepsilon^8)} = \frac{\mathcal{O}_M(\varepsilon^9)}{2 + 2\varepsilon_i\sigma_{1,i} + \mathcal{O}_M(\varepsilon^8)} = \mathcal{O}_M(\varepsilon^9)$$

and

$$\widehat{r}_i = \frac{2(\varepsilon_i^3 + \varepsilon_i^4\sigma_{1,i})\mathcal{O}_M(r)}{|2 + 2\varepsilon_i\sigma_{1,i} + \mathcal{O}_M(\varepsilon^8)|\ |2 + 2\varepsilon_i\sigma_{1,i} + \mathcal{O}_M(\varepsilon^2 r)|} = \mathcal{O}_M(\varepsilon^3 r). \quad \square$$

The convergence analysis of inclusion methods (9) with the two-point correction (10) relies on the following assertion which is a special case of Theorem 3 given in [20]:

Theorem 4.1. *Given the error-recursion*

$$v_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k}\left(v_j^{(m)}\right)^{t_{ij}}, \quad (i \in \mathbb{I}_k;\ m \ge 0), \tag{22}$$

where $t_{ij} \ge 0,\ \alpha_i > 0,\ 1 \le i, j \le k$, *and* $v_i^{(m)} = \varepsilon_i^{(m)}$ *or* $v_i^{(m)} = r_i^{(m)}$. *Denote the matrix of exponents appearing in (22) with* T_k, *that is* $T_k = [t_{ij}]_{k\times k}$. *If the non-negative matrix* T_k *has the spectral radius* $\rho(T_k) > 1$ *and a corresponding eigenvector* $\boldsymbol{x}_\rho > 0$, *then all sequences* $\{v_i^{(m)}\}$ $(i \in \mathbb{I}_k)$ *have the R-order at least* $\rho(T_k)$.

The matrix $T_k = [t_{ij}]$ will be called the R-*matrix* since it is directly associated with the R-order of convergence. Further, let $O_R(IM, \boldsymbol{\zeta})$ denote the R-order of convergence of an iterative method IM with the limit point $\boldsymbol{\zeta}$. For the inclusion method (16) we can state the following theorem.

Theorem 4.2. *If $Z_1^{(0)}, \ldots, Z_n^{(0)}$ are sufficiently close initial approximations to the distinct zeros $\zeta_1, \ldots, \zeta_n$, then the lower bound of the R-order of convergence of the interval method (16) is nine.*

Proof. For simplicity, as quite common in this type of analysis, we adopt the relation $1 > |\varepsilon^{(0)}| = r^{(0)} > 0$, which represents the "worst case" model. This assumption has no influence on the final result of the limit process which we apply in order to obtain the lower bound for the R-order of convergence. By virtue of Lemma 4.2 we notice sequences behaving in the following manner

$$\varepsilon^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^9, \qquad r^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^3 r^{(m)}.$$

Based on these relations and with accordance to Theorem 4.1, we form the R-matrix

$$T_2 = \begin{bmatrix} 9 & 0 \\ 3 & 1 \end{bmatrix}$$

with the spectral radius $\rho(T_2) = 9$ and the corresponding eigenvector $\boldsymbol{x}_\rho = (8, 3) > 0$. Hence, according to Theorem 4.1, we obtain

$$O_R((16), \boldsymbol{\zeta}) \geq \rho(T_2) = 9. \quad \square$$

5. Single-step methods

The convergence of the method (8), can be accelerated by applying the Gauss–Seidel approach. Thus, we use the already calculated circular approximations in the same iteration. In this manner we obtain the single-step method

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{B_i^{(m)}}, \quad (i \in \mathbb{I}_n) \tag{23}$$

$$B_i^{(m)} = \left[H(z_i^{(m)})\right]^{-1} - \frac{1}{2}N(z_i^{(m)})\left[S_{1,i}^2(Z^{(m+1)}, Z^{(m)}) + S_{2,i}(Z^{(m+1)}, Z^{(m)})\right].$$

The R-order of the method (23) is at least $3 + x_n$, where $x_n > 1$ is the unique positive root of the equation $x^n - x - 3 = 0$. For the proof see [4].

Similarly, applying the same Gauss–Seidel procedure to the method (9) we obtain a single-step method

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{BC_i^{(m)}}, \quad (i \in \mathbb{I}_n), \tag{24}$$

$$BC_i^{(m)} = \left[H(z_i^{(m)})\right]^{-1} - \frac{1}{2}N(z_i^{(m)})\left[S_{1,i}^2(Z^{(m+1)}, Z_C^{(m)}) + S_{2,i}(Z^{(m+1)}, Z_C^{(m)})\right].$$

The interval method (24) with correction C in the form of Newton's and Halley's iteration functions was examined in [3]. It was proved that $O_R((9)_N) \in (5, 6.646)$ in the case of Newton's correction and $O_R((9)_H) \in (6, 7.855)$ when the Halley's correction was applied.

Let us examine now, the case when the correction C is the two-point correction involved in (10).

It is very difficult to find the R-order of convergence of the single step methods (24) for a general n. In order to do that, one has to handle $2n$ mutually dependent sequences of centers and radii of produced disks, which is a very difficult task. Also, the number of zeros n is involved as a parameter. However, we can estimate easily the limit bounds of the R-order taking the limit cases $n = 2$ and very large n.

Since the convergence rate of a single-step method becomes almost the same to the one of the corresponding total-step method when the polynomial degree is very large, we obtain

$$O_R\left((24)_{TPC}, \boldsymbol{\zeta}\right) > O_R((9)_{TPC}, \boldsymbol{\zeta}) \geq 9.$$

Consider now the single-step methods (24) for $n = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)} < 1$ (the "worst case" model). After an extensive calculation we derive the following estimates:

$$|\hat{\varepsilon}_1| \sim |\varepsilon_1|^3|\varepsilon_2|^6, \qquad |\hat{\varepsilon}_2| \sim |\varepsilon_1|^3|\varepsilon_2|^9, \qquad \hat{r}_1 \sim |\varepsilon_1|^3 r_2, \qquad \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2.$$

The corresponding R-matrices and their spectral radii along with the relevant eigenvector for the method (24) are

$$T_4 = \begin{bmatrix} 3 & 6 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \qquad \begin{array}{l} \rho(T_4) = 11.19615, \\ \boldsymbol{x}_\rho = (1.436, 1.962, 0.474, 1.) > 0. \end{array}$$

According to the previous results we can state the following assertion:

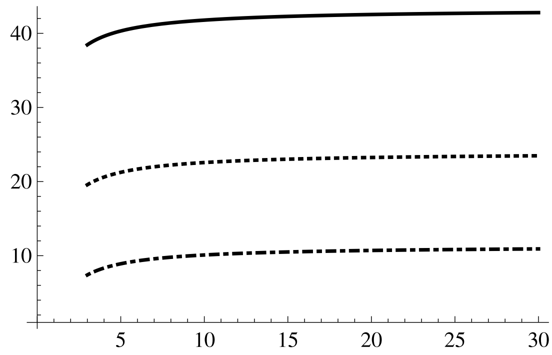

Fig. 1. Ratios of the computational efficiency.

Theorem 5.1. *The range of the lower bounds of the R-order of convergence of the single-step method (24) with the two-point correction is*

$$O_R((24)_{\text{TPC}}, \zeta) \in (9, 11.196).$$

6. Computational aspects

In this section we compute computational efficiency of the Halley-like method (8), and the Halley-like method with Newton's correction $(9)_N$, Halley's correction $(9)_H$ and the two-point correction $(9)_{TPC}$.

The efficiency of an iterative method (IM) can be estimated by the efficiency index

$$E(IM) = \frac{\log r}{d}, \tag{25}$$

where r is the R-order of convergence of the iterative method (IM) and d is its computational cost (see [4,8,21]). We emphasize that the rank list of methods obtained by formula (25) mainly well matches the real *CPU* time.

To obtain computational cost d we will use the number of arithmetic operations per iteration taken with certain weights depending on the execution time of operations

$$d = w_{as}AS + w_m M + w_d D + w_r R,$$

where AS, M, D and R are the number of additions+subtractions, multiplications, divisions and extractions of a root and w_{as}, w_m, w_d and w_r are corresponding weights.

Weights w_{as}, w_m, w_d and w_r in (25) are determined according to the estimation of the complexity of the basic operations in multiple-precision arithmetic. We assume that the floating point number representation is used with a binary fraction of b bits. According to the results given in [22] the execution time for additions and subtractions t_{AS} is $\mathcal{O}(b)$. Using Schönhage–Strassen multiplication we obtain that the execution time for multiplication t_m is $\mathcal{O}(b \log b \log\log b)$. For division, it is scaled by $t_d = 3.5t_m$, while for the extraction of a root we have $t_r = 4t_m$. We chose the weights w_{as}, w_m, w_d and w_r to be proportional to $t_{AS}(b)$, $t_m(b)$, $t_d(b)$ and $t_r(b)$ for a 128-bit architecture ($b = 128$).

Complex polynomials P_n with real or complex zeros are considered. The number of basic complex operations is reduced to the number of operations in real arithmetic. The obtained values, as functions of the polynomial degree n, are given in Table 2.

Applying (25) and data from Table 2 we calculated the percent ratios

$$\rho\big((9)_{TPC}, (X)\big)(n) = \left(\frac{E\big((9)_{TPC}, n\big)}{E(X, n)} - 1\right) \cdot 100 \quad (\text{in } \%),$$

where (X) is one of the methods (8), $(9)_N$ and $(9)_H$. The ratios $\rho\big((9)_{TPC}, (X)\big)(n)$ present the increase of computational efficiency of the inclusion method $(9)_{TPC}$ in the relation to the methods (8), $(9)_N$ and $(9)_H$. These relations are graphically presented in Fig. 1 as functions of the polynomial degree n, where the ratio $\rho\big((9)_{TPC}, (8)\big)(n)$ is displayed by full line, the ratio $\rho\big((9)_{TPC}, (9)_N\big)(n)$ by dotted line, while the ratio $\rho\big((9)_{TPC}, (9)_H\big)(n)$ is displayed by dot-dashed line.

From Fig. 1 we can observe that the interval method $(9)_{TPC}$ is the most efficient. The obtained percentage improvement depends on the weights w_{as}, w_m, w_d and w_r in (25). Different choice of weights gives slightly different outcomes, but the average ratios of computational efficiency of the considered methods lead to the same conclusion.

The presented total step methods (8), $(9)_N$, $(9)_H$ and $(9)_{TPC}$ and single step methods (23), $(24)_N$, $(24)_H$ and $(24)_{TPC}$ have been tested in solving many polynomial equations. To provide the enclosure of the zeros in the fourth and fifth iteration that produce very small disks, we used the programming package *Mathematica* with multi-precision arithmetic.

Table 3
Radii of total step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(8)	7.62(−2)	2.21(−7)	1.11(−32)	9.07(−134)	2.79(−538)	4.0016
$(9)_N$	6.14(−2)	4.70(−9)	3.15(−44)	1.49(−219)	8.15(−1096)	4.9979
$(9)_H$	6.22(−2)	6.29(−11)	1.62(−64)	1.17(−385)	3.30(−2311)	5.9960
$(9)_{TPC}$	6.20(−2)	3.88(−14)	3.17(−123)	5.43(−1107)	9.63(−9963)	9.0019

Table 4
Radii of single step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(23)	1.52(−2)	1.47(−10)	1.81(−43)	6.45(−178)	1.51(−718)	4.0211
$(24)_N$	1.74(−2)	7.35(−10)	1.29(−49)	1.63(−255)	5.89(−1325)	5.1940
$(24)_H$	1.57(−2)	9.62(−12)	1.03(−71)	6.51(−449)	2.97(−2731)	6.0508
$(24)_{TPC}$	1.57(−2)	6.03(−15)	7.61(−131)	5.73(−1179)	1.12(−10 638)	9.0254

It is of interest to check the convergence rate of the proposed interval methods in practical implementation and compare it to the theoretical order given in the presented convergence theorems. For this reason, we calculated the so-called *computational order of convergence*, briefly denoted by $COC^{(m)}$, for the m-the iteration:

$$COC^{(m)} = \frac{\log\left[r^{(m)}/r^{(m-1)}\right]}{\log\left[r^{(m-1)}/r^{(m-2)}\right]}, \tag{26}$$

where, as above, $r^{(m)}$ is the maximal radii of inclusion disks produced at the m-th iteration. The values of $COC^{(m)}$ are given in the last columns of the tables presented below.

Remark 6.1. Formula (26) is an adaptation of a Jay's formula presented in [23], which gives good results if the accuracies of root approximations are of the same order. If approximations are not of same quality (for example, in the case of single-step methods), then a certain difference between the computational order of convergence and the theoretical order can appear, but usually to acceptable extent from a practical point of view.

Example 1. To find the circular inclusion approximations to the zeros of the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

we implemented the interval methods (8), (9) (with Newton's, Halley's and two point correction), (23) and (24) (with Newton's, Halley's and two point correction). The exact zeros of P are $\zeta_1 = -3$, $\zeta_{2,3} = \pm 1$, $\zeta_{4,5} = \pm 2i$, $\zeta_{6,7} = -2 \pm i$, $\zeta_{8,9} = 2 \pm i$. The initial disks were selected to be $\boldsymbol{Z}_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$z_1^{(0)} = -3.1 + 0.2i, \quad z_2^{(0)} = -1.2 - 0.1i, \quad z_3^{(0)} = 1.2 + 0.1i,$$
$$z_4^{(0)} = 0.2 - 2.1i, \quad z_5^{(0)} = 0.2 + 1.9i, \quad z_6^{(0)} = -1.8 + 1.1i,$$
$$z_7^{(0)} = -1.8 - 0.9i, \quad z_8^{(0)} = 2.1 + 1.1i, \quad z_9^{(0)} = 1.8 - 0.9i.$$

The maximal radii of the inclusion disks produced in the first five iterative steps, are given in Tables 3 and 4, where the denotation $A(-q)$ means $A \times 10^{-q}$.

Example 2. We have applied the same inclusion methods in order to find the circular inclusion approximations to the zeros of the polynomial

$$\begin{aligned} P(z) = {} & z^{20} + 12z^{19} + 80z^{18} + 360z^{17} + 1356z^{16} + 4512z^{15} + 13\,440z^{14}35\,520z^{13} + 84\,976z^{12} + 192\,192z^{11} \\ & + 416\,000z^{10} + 574\,080z^9 - 153\,024z^8 - 3\,283\,968z^7 - 8\,048\,640z^6 - 15\,452\,160z^5 \\ & - 20\,317\,184z^4 - 15\,925\,248z^3 - 38\,010\,880z^2 - 68\,812\,800z - 73\,728\,000. \end{aligned}$$

The exact zeros of P are $\zeta_{1,2} = 1 \pm i$, $\zeta_{3,4} = 1 \pm 3i$, $\zeta_{5,6} = 2 \pm 2i$, $\zeta_{7,8} = \pm 2$, $\zeta_{9,10} = \pm 2i$, $\zeta_{11,12} = -1 \pm i$, $\zeta_{13,14} = -1 \pm 3i$, $\zeta_{15,16} = -2 \pm 2i$, $\zeta_{17,18} = -3 \pm i$, $\zeta_{19,20} = -3 \pm 3i$. The initial disks were selected to be $\boldsymbol{Z}_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$z_1^{(0)} = 0.9 + 1.2i, \quad z_2^{(0)} = 0.8 - 1.1i, \quad z_3^{(0)} = 0.9 + 2.9i,$$
$$z_4^{(0)} = 1.2 - 3.1i, \quad z_5^{(0)} = 2.2 + 2.1i, \quad z_6^{(0)} = 2.1 - 2.2i,$$
$$z_7^{(0)} = -1.8 + 0.1i, \quad z_8^{(0)} = 1.9 - 0.1i, \quad z_9^{(0)} = -0.1 + 2.2i,$$

Table 5
Radii of total step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(8)	1.21(−1)	6.62(−7)	1.87(−29)	4.78(−125)	7.62(−506)	3.9836
$(9)_N$	1.32(−1)	2.65(−7)	1.37(−37)	1.55(−188)	5.93(−941)	4.9847
$(9)_H$	1.24(−1)	3.00(−9)	1.50(−56)	3.21(−338)	1.12(−2026)	5.9945
$(9)_{TPC}$	1.28(−1)	3.77(−10)	6.91(−87)	2.51(−773)	3.89(−6952)	9.0012

Table 6
Radii of single step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(23)	1.11(−1)	9.37(−8)	5.48(−33)	8.07(−135)	1.65(−546)	4.0428
$(24)_N$	1.11(−1)	2.76(−8)	5.26(−42)	9.38(−212)	4.83(−1067)	5.0386
$(24)_H$	1.06(−1)	6.28(−10)	5.80(−61)	3.61(−367)	6.02(−2217)	6.0410
$(24)_{TPC}$	1.09(−1)	2.39(−11)	1.48(−95)	3.33(−826)	3.33(−7434)	9.0440

Table 7
Radii of total step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(8)	7.96(−2)	1.19(−6)	5.16(−29)	2.02(−119)	8.37(−485)	4.0416
$(9)_N$	1.14(−1)	3.78(−7)	1.50(−35)	7.35(−178)	1.01(−887)	4.9882
$(9)_H$	1.17(−1)	2.65(−8)	8.60(−53)	5.79(−317)	3.36(−1900)	5.9932
$(9)_{TPC}$	1.07(−1)	2.60(−8)	1.11(−72)	4.33(−648)	1.86(−5820)	8.9890

$$z_{10}^{(0)} = 0.2 - 1.9i, \quad z_{11}^{(0)} = -0.8 + 1.1i, \quad z_{12}^{(0)} = -1.1 - 1.2i,$$
$$z_{13}^{(0)} = -0.9 + 2.9i, \quad z_{14}^{(0)} = -0.9 - 2.9i, \quad z_{15}^{(0)} = -2.1 + 2.2i,$$
$$z_{16}^{(0)} = -2.2 - 2.1i, \quad z_{17}^{(0)} = -2.9 + 1.2i, \quad z_{18}^{(0)} = -2.9 - 1.1i,$$
$$z_{19}^{(0)} = -2.9 + 2.9i, \quad z_{20}^{(0)} = -2.9 - 2.9i.$$

The maximal radii of the inclusion disks produced in the first five iterative steps, are given in Tables 5 and 6.

Example 3. We have applied the same inclusion methods in order to find the circular inclusion approximations to the zeros of the polynomial

$$\begin{aligned} P(z) = {} & z^{25} - 15z^{24} + 87z^{23} - 231z^{22} + 398z^{21} - 2904z^{20} + 20\,472z^{19} - 66\,816z^{18} + 52\,918z^{17} + 403\,206z^{16} \\ & - 1\,763\,478z^{15} + 1\,980\,534z^{14} + 12\,830\,648z^{13} - 76\,002\,444z^{12} + 202\,129\,932z^{11} - 356\,907\,996z^{10} \\ & + 523\,871\,353z^{9} - 342\,789\,039z^{8} - 1\,845\,963\,753z^{7} + 8\,666\,158\,809z^{6} - 17\,149\,936\,318z^{5} \\ & + 14\,381\,171\,196z^{4} + 1\,645\,576\,740z^{3} - 8\,311\,164\,300z^{2} + 16\,613\,181\,000z - 13\,962\,780\,000. \end{aligned}$$

The exact zeros of P are $\zeta_{1,2} = \pm 1$, $\zeta_{3,4} = \pm i$, $\zeta_{5,6} = 1 \pm 2i$, $\zeta_{7,8} = -1 \pm 2i$, $\zeta_{9,10} = 2 \pm i$, $\zeta_{11,12} = -3 \pm i$, $\zeta_{13,14} = \pm 3i$, $\zeta_{15} = 3$, $\zeta_{16} = -2$, $\zeta_{17,18} = 3 \pm 2i$, $\zeta_{19,20} = 4 \pm i$, $\zeta_{21,22} = 2 \pm 3i$, $\zeta_{23,24} = -3 \pm 3i$, $\zeta_{25} = 4$. The initial disks were selected to be $\boldsymbol{Z}_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$z_{1}^{(0)} = 1.1 + 0.2i, \quad z_{2}^{(0)} = -1.2 - 0.1i, \quad z_{3}^{(0)} = 0.2 + 1.1i,$$
$$z_{4}^{(0)} = -0.2 - 1.1i, \quad z_{5}^{(0)} = 1.2 + 2.1i, \quad z_{6}^{(0)} = 1.1 - 2.1i,$$
$$z_{7}^{(0)} = -1.2 + 1.9i, \quad z_{8}^{(0)} = -1.2 - 1.9i, \quad z_{9}^{(0)} = 2.1 + 1.2i,$$
$$z_{10}^{(0)} = 2.2 - 1.1i, \quad z_{11}^{(0)} = -3.2 + 1.1i, \quad z_{12}^{(0)} = -3.2 - 1.1i,$$
$$z_{13}^{(0)} = 0.1 + 2.9i, \quad z_{14}^{(0)} = 0.1 - 2.9i, \quad z_{15}^{(0)} = 2.9 + 0.1i,$$
$$z_{16}^{(0)} = -2.2 - 0.1i, \quad z_{17}^{(0)} = 2.9 + 2.1i, \quad z_{18}^{(0)} = 3.2 - 2.1i,$$
$$z_{19}^{(0)} = 3.9 + 1.1i, \quad z_{20}^{(0)} = 3.9 - 1.1i, \quad z_{21}^{(0)} = 2.2 + 2.9i,$$
$$z_{22}^{(0)} = 2.1 - 3.1i, \quad z_{23}^{(0)} = -3.2 + 2.9i, \quad z_{24}^{(0)} = -3.2 - 2.9i,$$
$$z_{25}^{(0)} = 3.9 - 0.1i.$$

The maximal radii of the inclusion disks produced in the first five iterative steps, are given in Tables 7 and 8.

Table 8
Radii of single step methods.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$COC^{(5)}$
(23)	4.80(−2)	6.80(−8)	1.52(−35)	1.82(−148)	2.73(−598)	3.9835
$(24)_N$	6.14(−2)	3.73(−8)	2.32(−42)	1.62(−216)	3.73(−1095)	5.0451
$(24)_H$	6.90(−2)	4.35(−9)	1.96(−55)	4.30(−330)	3.18(−1999)	6.0771
$(24)_{TPC}$	6.96(−2)	5.78(−9)	3.33(−74)	4.24(−658)	9.16(−6003)	9.1535

It is evident from Tables 3–8 and a number of tested polynomial equations that the presented methods behave very fast convergence. Among them we observe that the method $(9)_{TPC}$ and its single-step variant $(24)_{TPC}$ are the most efficient.

From the last columns of Tables 3–8 we can conclude that obtained computational order of convergence for the proposed methods (8), $(9)_N$, $(9)_H$, $(9)_{TPC}$, (23), $(24)_N$, $(24)_H$ and $(24)_{TPC}$ well coincides with the theoretical order of convergence. This conclusion especially holds for the total-step methods (Tables 3, 5 and 7). The computational order of convergence of single-step methods (Tables 4, 6 and 8) gives relatively good agreement with the theoretical order, acceptable for a practical purpose. Some deviations happened due to the impossibility of precise description of approximations of different accuracy, which has been stressed in Remark 6.1. Actually, the computational order of a single-step method is close to the order of the corresponding total-step method, which could be expected since the formula (26) follows disks of maximal radii. However, the user can get a good estimate of convergence rate, most frequently acceptable for practical purpose.

Acknowledgments

Authors would like to thank reviewers for their valuable comments and guidance that improved the presentation.

References

[1] M.S. Petković, On Halley-like algorithms for the simultaneous approximation of polynomial complex zeros, SIAM J. Numer. Math. 26 (1989) 740–763.
[2] X. Wang, S. Zheng, A family of parallel and interval iterations for finding simultaneously all roots of a polynomial with rapid convergence (I), J. Comput. Math. 4 (1984) 70–76.
[3] D.M. Milošević, The accelerated Halley-like method for the inclusion of polynomial zeros, in: N. Krejić, Z. Lužanin (Eds.), Proceedings of the XVI Conference on Applied Mathematics, Budva 2004, Institute of Mathematics, Novi Sad, 2006, pp. 83–91.
[4] M.S. Petković, Iterative Methods for Simultaneous Inclusion of Polynomial Zeros, Springer-Verlag, Berlin, Heidelberg, New York, 1989.
[5] M.S. Petković, Halley-like method with corrections for the inclusion of polynomial zeros, Computing 62 (1999) 69–88.
[6] M.S. Petković, D.M. Milošević, Improved Halley-like methods for the inclusion of polynomial zeros, Appl. Math. Comput. 169 (2005) 417–436.
[7] M.S. Petković, D.M. Milošević, Higher order methods for the inclusion of multiple zeros of polynomials, Reliab. Comput. 15 (2011) 91–108.
[8] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, Multipoint Methods for Solving Nonlinear Equations, Elsevier, Amsterdam, Boston, Heidelberg, 2013.
[9] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, Multipoint methods for solving nonlinear equations: A survey, Appl. Math. Comput. 226 (2014) 635–660.
[10] I. Gargantini, P. Henrici, Circular arithmetic and the determination of polynomial zeros, Numer. Math. 18 (1972) 305–320.
[11] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, New York, 1983.
[12] M.S. Petković, L.D. Petković, Complex Interval Arithmetic and its Applications, Wiley-VCH, Berlin, Weinheim, New York, 1998.
[13] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
[14] A.W.M. Nourein, An improvement on two iteration methods for simultaneous determination of the zeros of a polynomial, Int. J. Comput. Math. 6 (1977) 241–252.
[15] C. Carstensen, M.S. Petković, An improvement of Gargantini's simultaneous inclusion method for polynomial roots by Schroeder's correction, Appl. Numer. Math. 13 (1994) 453–468.
[16] M.S. Petković, C. Carstensen, On some improved inclusion methods for polynomial roots with Weierstrass' correction, Comput. Appl. Math. 25 (1993) 59–67.
[17] I. Gargantini, Further application of circular arithmetic: Schröder-like algorithms with error bound for finding zeros of polynomials, SIAM J. Numer. Anal. 15 (1978) 497–510.
[18] I. Gargantini, The numerical stability of simultaneous iteration via square-rooting, Comput. Math. Appl. 5 (1979) 25–31.
[19] M.S. Petković, L.V. Stefanović, The numerical stability of the generalised root iterations for polynomial zeros, Comput. Math. Appl. 10 (1984) 97–106.
[20] J. Herzberger, L. Metzner, On the Q-order and R-order of convergence for coupled sequences arising in iterative numerical processes, in: G. Alefeld, J. Herzberger (Eds.), Numerical Methods and Error Bounds, in: Mathematical Research, vol. 89, Akademie Verlag, Berlin, 1996, pp. 120–131.
[21] J.M. McNamee, Numerical Methods for Roots of Polynomials, Part I, Elsevier, Amsterdam, 2007.
[22] R. Brent, P. Zimmermann, Modern Computer Arithmetic, Cambridge University Press, Cambridge, 2011.
[23] L.O. Jay, A note on Q-order of convergence, BIT 41 (2001) 422–429.