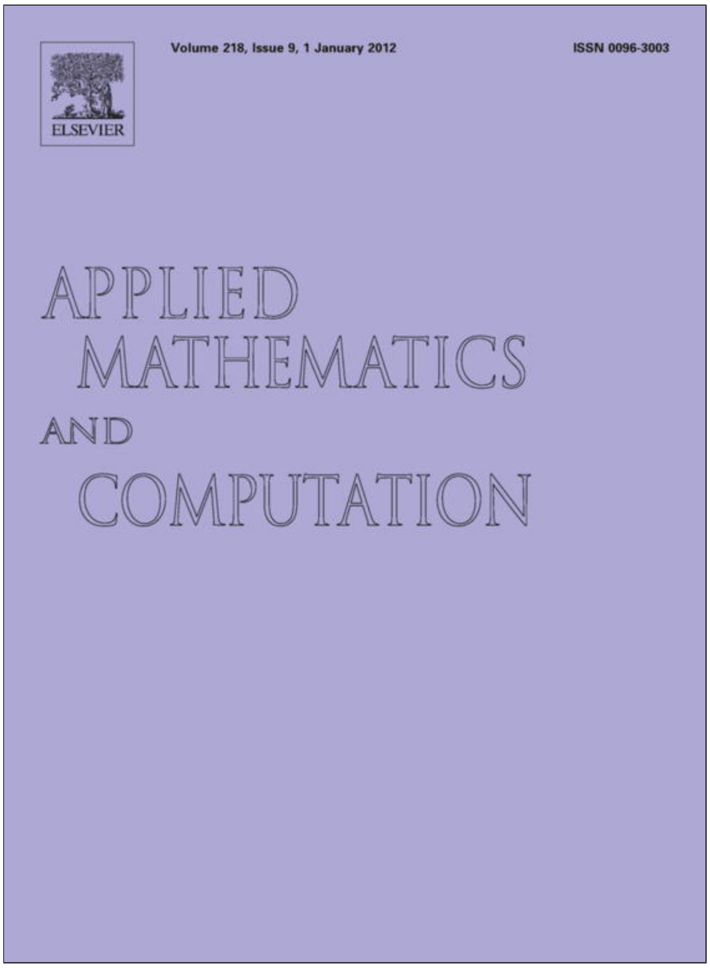
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Three-point methods with and without memory for solving nonlinear equations

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ABSTRACT

A new family of three-point derivative free methods for solving nonlinear equations is presented. It is proved that the order of convergence of the basic family without memory is eight requiring four function-evaluations, which means that this family is optimal in the sense of the Kung–Traub conjecture. Further accelerations of convergence speed are attained by suitable variation of a free parameter in each iterative step. This self-accelerating parameter is calculated using information from the current and previous iteration so that the presented methods may be regarded as the methods with memory. The self-correcting parameter is calculated applying the secant-type method in three different ways and Newton's interpolatory polynomial of the second degree. The corresponding *R*-order of convergence is increased from 8 to $4(1+\sqrt{5}/2) \approx 8.472$, 9, 10 and 11. The increase of convergence order is attained without any additional function calculations, providing a very high computational efficiency of the proposed methods with memory. Another advantage is a convenient fact that these methods do not use derivatives. Numerical examples and the comparison with existing three-point methods are included to confirm theoretical results and high computational efficiency.

1. Introduction

The most efficient existing root-solvers are based on multipoint iterations since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. The upper bound of order of multiple methods was discussed in [8] by Kung and Traub who conjectured that the order of convergence of any multipoint method without memory, consuming $n+1$ function evaluations per iteration, cannot exceed the bound 2^n (called *optimal order*). This hypothesis has not been proved yet but it turned out that all existing methods constructed at present support the Kung–Traub conjecture.

In this paper we derive a new family of three-point methods of order eight, requiring four function evaluations per iteration. This means that the proposed family supports the Kung–Traub conjecture, too. Besides, this family does not use any derivative of a function f whose zeros are sought, which is another advantage since it is preferable to avoid calculations of derivatives of f in many practical situations.

Bearing in mind that derivative free higher-order multipoint methods without memory were already derived in the literature, see [8,19], the proposed family of three-point methods could be regarded as a competitive contribution to the topic, but without particular advances. However, using an old idea by Traub [16], recently extended in [12], we improved this basic family without memory and constructed the corresponding family of three-point methods with memory. We show that the order of convergence of the new family can be considerably increased by varying a free parameter in each iterative step. The

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significant increase of convergence speed is achieved without additional function evaluations. This means that the proposed methods with memory possess a very high computational efficiency, which is the main advantage of these methods compared with existing multi-point methods.

This paper is organized as follows. In Section 2 we describe a derivative free family of two-point methods of order four. This family is the base for constructing a new family of three-point methods of optimal order eight, which also does not use derivatives (Section 3). The proposed eight order family depends on a real parameter that can be recursively calculated during the iterative process in order to accelerate the convergence. In Section 4 we present four approaches for calculation of this varying parameter, called *self-accelerating parameter*, relied on the secant-type approach in three different ways and Newton's interpolatory polynomial of second degree. These accelerating techniques use information from the current and the previous iterative step, defining in this way three-point methods with memory. It is shown in Section 5 that the R-order of the corresponding methods with memory is increased from 8 (the basic family without memory) to $2(2+\sqrt{5})\approx 8.472$, 9, 10 and 11, depending on the accelerating technique. Numerical examples and the comparison with existing three-point methods are given in Section 6 to confirm theoretical results and to demonstrate very fast convergence and a high computational efficiency of the proposed methods.

2. Derivative free two-point methods

Let α be a simple real zero of a real function $f: D\subset \boldsymbol{R}\to\boldsymbol{R}$ and let x_0 be an initial approximation to α. As in the case of the Kung–Traub family of derivative free methods [8], we start with the derivative free method

$$x_{k+1}=x_k-\frac{\gamma f(x_k)^2}{f(x_k+\gamma f(x_k))-f(x_k)}\quad (k=0,1,\ldots) \tag{1}$$

of Steffensen's type with quadratic convergence (see [16, p. 185]), where γ is a real constant.

Introduce the abbreviations

$$\varepsilon_k=x_k-\alpha,\quad c_k=\frac{f^{(k)}(\alpha)}{k!f'(\alpha)}\quad (k=2,3,\ldots).$$

Let

$$\varphi(x)=\frac{f(x+\gamma f(x))-f(x)}{\gamma f(x)}, \tag{2}$$

be a function that appears in the Steffensen-like method (1). The following derivative free family of two-point iterative methods was derived in [13],

$$\begin{cases} y_k=x_k-\frac{f(x_k)}{\varphi(x_k)},\\ x_{k+1}=y_k-h(u_k,v_k)\frac{f(y_k)}{\varphi(x_k)} \end{cases}\quad (k=0,1,\ldots), \tag{3}$$

where

$$u_k=\frac{f(y_k)}{f(x_k)},\qquad v_k=\frac{f(y_k)}{f(x_k+\gamma f(x_k))}$$

and h is a two-valued function that satisfies the conditions

$$h(0,0)=h_u(0,0)=h_v(0,0)=1,\quad h_{vv}(0,0)=2,\quad |h_{uu}(0,0)|<\infty,\quad |h_{uv}(0,0)|<\infty. \tag{4}$$

Here the subscript indices denote corresponding partial derivatives of h.

If x_0 is an initial approximation sufficiently close to the zero α of f, it was proved in [13] that the family of two-point methods (3) is of order four and the error relation

$$\varepsilon_{k+1}=x_{k+1}-\alpha=-c_2(1+\gamma f'(\alpha))^2\left[c_3+c_2^2(-4+h_{uu}(0,0)/2+h_{uv}(0,0)+(h_{uu}(0,0)/2-1)\gamma f'(\alpha))\right]\varepsilon_k^4+O(\varepsilon_k^5). \tag{5}$$

holds.

Remark 1. Considering the double Newton scheme

$$\begin{cases} y_k=x_k-\frac{f(x_k)}{f'(x_k)},\\ x_{k+1}=y_k-\frac{f(y_k)}{f'(y_k)} \end{cases}\quad (k=0,1,\ldots) \tag{6}$$

and (3), we see that $\varphi(x)$ is an approximation to the first derivative $f'(x)$ in (6) assuming that $|f(x)|$ is small enough. The derivative $f'(y)$ in the second step of (6) is approximated by $\varphi(x)/h(u,v)$, where $h(u,v)$ satisfies the conditions (4).

Henceforth we will consider that the function $h=h(u,v)$ satisfies the conditions (4) without being cited. Several simple forms of the function h are given below:

(1) $h(u,v) = \frac{1+u}{1-v}$;
(2) $h(u,v) = \frac{1}{(1-u)(1-v)}$;
(3) $h(u,v) = 1 + u + v + v^2$;
(4) $h(u,v) = 1 + u + v + (u+v)^2$;
(5) $h(u,v) = u + \frac{1}{1-v}$.

Note that the function $h(u,v) = \frac{1}{(1-u)(1-v)}$ gives the Kung–Traub method

$$\begin{cases} y_k = x_k - \frac{\gamma f(x_k)^2}{f(x_k+\gamma f(x_k)) - f(x_k)}, \\ x_{k+1} = y_k - \frac{f(y_k) f(x_k+\gamma f(x_k))}{(f(x_k+\gamma f(x_k)) - f(y_k)) f[x_k, y_k]}, \end{cases} \quad (k = 0, 1, \ldots), \tag{7}$$

where $f[x,y] = (f(x) - f(y))/(x - y)$ denotes a divided difference. This method is obtained as a special case of Kung–Traub's family of derivative free methods presented in [8].

3. A new family of three-point methods

Now we construct a family of three-point methods relied on the two-step family (3). We start from a three-step scheme where the first two steps are given by (3), and the third step is Newton's method, that is,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{\varphi(x_k)}, \\ z_k = y_k - h(u_k, v_k) \frac{f(y_k)}{\varphi(x_k)}, \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}. \end{cases} \tag{8}$$

The iterative scheme (8) is inefficient since it requires five function evaluations. For this reason, the derivative $f'(z_k)$ in the third step of (8) should be substituted by a suitable approximation in such way that (i) only available data, not including calculation of derivatives, are used and (ii) the order of convergence of the new iterative three-step scheme is at least eight consuming four function evaluations. To provide these requirements, we apply Newton's interpolatory polynomial of degree three at the points $w_k = x_k + \gamma f(x_k)$, x_k, y_k and z_k, that is,

$$N_3(t) = f(z_k) + f[z_k, y_k](t - z_k) + f[z_k, y_k, x_k](t - z_k)(t - y_k) + f[z_k, y_k, x_k, w_k](t - z_k)(t - y_k)(t - x_k). \tag{9}$$

It is obvious that $N_3(z_k) = f(z_k)$. Differentiating (9) and setting $t = z_k$, we obtain

$$N_3'(z_k) = f[z_k, y_k] + f[z_k, y_k, x_k](z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k). \tag{10}$$

Substituting $f'(z_k) \approx N_3'(z_k)$ in (8) we state a new family of three-point methods free of derivatives,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{\varphi(x_k)}, \\ z_k = y_k - h(u_k, v_k) \frac{f(y_k)}{\varphi(x_k)}, \quad (k = 0, 1, \ldots), \\ x_{k+1} = z_k - \frac{f(z_k)}{f[z_k, y_k] + f[z_k, y_k, x_k](z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k)}, \end{cases} \tag{11}$$

where φ is defined by (2) and h is a two-valued weight function that satisfies (4). N_3' given by (10) (that is, the denominator of (11)) can be easily calculated by the five-step algorithm:

1° $R_1 = f[z, y] = \frac{f(z) - f(y)}{z - y}$;
2° $R_2 = f[y, x] = \frac{f(y) - f(x)}{y - x}$;
3° $R_3 = f[x, w] = \frac{f(x) - f(w)}{x - w}$;
4° $R_4 = f[z, y, x] = \frac{R_2 - R_1}{x - z}$;
5° $N_3'(z) = R_1 + R_4(z - y) + \left(\frac{R_3 - R_2}{w - y} - R_4\right) \frac{(z - y)(z - x)}{w - z}$.

Now we state the following convergence theorem for the family (11).

Theorem 1. *If an initial approximation x_0 is sufficiently close to the zero α of f and the weight function h satisfies the conditions (4), then the convergence order of the family of three-point methods (11) is equal to eight.*

Proof. Let N_m be the Newton interpolation polynomial of degree m that interpolates a function f at $m + 1$ distinct interpolation nodes $t_0, t_1, \ldots, t_m$ contained in an interval I and the derivative $f^{(m+1)}$ is continuous in I. Then the error of the Newton interpolation is given by the well known formula

$$f(t) - N_m(t) = \frac{f^{(m+1)}(\xi)}{(m+1)!}\prod_{j=0}^{m}(t-t_j) \quad (\xi \in I). \tag{12}$$

For $m = 3$ we have from (12)

$$f(t) - N_3(t) = \frac{f^{(4)}(\xi)}{4!}(t-w_k)(t-x_k)(t-y_k)(t-z_k),$$

taking $t_0 = w_k$, $t_1 = x_k$, $t_2 = y_k$, $t_3 = z_k$. Hence

$$f'(z_k) - N_3'(z_k) = [f'(t) - N_3'(t)]_{t=z_k} = \frac{f^{(4)}(\xi)}{4!}(z_k-w_k)(z_k-x_k)(z_k-y_k). \tag{13}$$

The errors at the first two steps of (11) are given by

$$\varepsilon_{k,y} := y_k - \alpha = c_2(1+\gamma f'(\alpha))\varepsilon_k^2 + O(\varepsilon_k^3) \quad (\text{see Traub [16, p. 185]}), \tag{14}$$

and

$$\varepsilon_{k,z} := z_k - \alpha = A_4(\alpha)\varepsilon_k^4 + O(\varepsilon_k^5) \quad (\text{see (5)}), \tag{15}$$

where A_4 is the asymptotic error constant of the fourth-order family (3) given by

$$A_4(\alpha) = -c_2(1+\gamma f'(\alpha))^2\left[c_3 + c_2^2(-4 + h_{uu}(0,0)/2 + h_{uv}(0,0) + (h_{uu}(0,0)/2 - 1)\gamma f'(\alpha))\right]$$

for a fixed constant γ ($\neq f'(\alpha)$). From (14) and (15) we find

$$z_k - w_k = O(\varepsilon_k), \quad z_k - x_k = O(\varepsilon_k), \quad z_k - y_k = O(\varepsilon_k^2). \tag{16}$$

Replacing the error differences given by (16) in (13), we obtain $f'(z_k) - N_3'(z_k) = O(\varepsilon_k^4)$ and hence

$$N_3'(z_k) = f'(z_k)(1 + O(\varepsilon_k^4)). \tag{17}$$

Substituting (17) in the third step of the iterative scheme (11) we find

$$x_{k+1} = z_k - \frac{f(z_k)}{N_3'(z_k)} = z_k - \frac{f(z_k)}{f'(z_k)(1+O(\varepsilon_k^4))} = z_k - \frac{f(z_k)}{f'(z_k)} + f(z_k)O(\varepsilon_k^4). \tag{18}$$

For Newton's method we have

$$z_k - \frac{f(z_k)}{f'(z_k)} - \alpha = c_2(z_k-\alpha)^2 + O((z_k-\alpha)^3) = c_2\varepsilon_{k,z}^2 + O\left(\varepsilon_{k,z}^3\right). \tag{19}$$

Also, observe that

$$f(z_k) = (z_k - \alpha)g(z_k) = \varepsilon_{k,z}g(z_k), \quad g(z_k) \neq 0 \quad \text{with} \quad g(z_k) \to g(\alpha) \quad \text{when} \quad z_k \to \alpha. \tag{20}$$

Taking into account (19) and (20), we find from (18)

$$\varepsilon_{k+1} = x_{k+1} - \alpha = c_2\varepsilon_{k,z}^2 + O\left(\varepsilon_{k,z}^3\right) + \varepsilon_{k,z}g(z_k)O(\varepsilon_k^4) = O(\varepsilon_k^8),$$

since $\varepsilon_{k,z} = O(\varepsilon_k^4)$. From the last error relation we conclude that the order of convergence of the family (11) is eight, which completes the proof of Theorem 1. □

Remark 2. The proof of Theorem 1 can also be derived using Taylor's series and symbolic computation in a computer algebra system (e.g., *Mathematica* or *Maple*) as performed, for example, in [15]. In this way we arrive at the error relation

$$\begin{aligned}\varepsilon_{k+1} = {} & \frac{c_2^2}{4}(1+\gamma f'(\alpha))^4\left[2c_3 + c_2^2(-8 + 2h_{uv}(0,0) + \gamma f'(\alpha)(h_{uu}(0,0)-2) + h_{uu}(0,0))\right] \\ & \times \left[2c_2c_3 - 2c_4 + c_2^3(-8 + 2h_{uv}(0,0) + \gamma f'(\alpha)(h_{uu}(0,0)-2) + h_{uu}(0,0))\right]\varepsilon_k^8 + O(\varepsilon_k^9).\end{aligned} \tag{21}$$

The error relations of the three-point methods (11) for particular forms (1)–(5) of h, given above, can be calculated from (21). The corresponding expressions are listed below:

$h(u,v) = 1 + u + v + v^2$ and $h(u,v) = u + 1/(1-v)$,
$\varepsilon_{k+1} = (1+\gamma f'(\alpha))^4 c_2^2\big(-c_3 + c_2^2(4+\gamma f'(\alpha))\big)\big(-c_2c_3 + c_4 + c_2^3(4+\gamma f'(\alpha))\varepsilon_k^8 + O(\varepsilon_k^9)\big)$.
$h(u,v) = (1+u)/(1-v)$,
$\varepsilon_{k+1} = (1+\gamma f'(\alpha))^4 c_2^2\big(-c_3 + c_2^2(3+\gamma f'(\alpha))\big(-c_2c_3 + c_4 + c_2^3(3+\gamma f'(\alpha))\big)\varepsilon_k^8 + O(\varepsilon_k^9)\big)$.
$h(u,v) = 1/((1-u)(1-v))$,
$\varepsilon_{k+1} = (1+\gamma f'(\alpha))^4 c_2^2\big(2c_2^2 - c_3\big)\big(2c_2^3 - c_2c_3 + c_4\big)\varepsilon_k^8 + O(\varepsilon_k^9)$.
$h(u,v) = 1 + u + v + (u+v)^2$,
$\varepsilon_{k+1} = (1+\gamma f'(\alpha))^4 c_2^2\big(c_2^2 - c_3\big)\big(c_2^3 - c_2c_3 + c_4\big)\varepsilon_k^8 + O(\varepsilon_k^9)$.

4. New families of three-point methods with memory

We observe from (5) and (21) that the order of convergence of the families (3) and (11) is respectively four and eight when $\gamma \neq -1/f'(\alpha)$. If we could provide that $\gamma = -1/f'(\alpha)$, it can be proved that the order of the families (3) and (11) would be 6 and 12, respectively. However, the value $f'(\alpha)$ is not available in practice and such acceleration of convergence is not possible. Instead of that, we could use an approximation $\bar{f}'(\alpha) \approx f'(\alpha)$, calculated by available information. Then, by setting $\gamma = -1/\bar{f}'(\alpha)$ in (11), we can achieve that the order of convergence of the modified methods exceeds eight without the use of any new function evaluations. We will see later that $\bar{f}'(\alpha)$ is calculated using information from the current and previous iteration, in other words, $\bar{f}'(\alpha)$ depends on the iteration index k. However, we omit the iteration index for simplicity.

In this paper we consider the following four methods for approximating $f'(\alpha)$:

(I) $\bar{f}'(\alpha) = \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$ (simple secant approach).
(II) $\bar{f}'(\alpha) = \frac{f(x_k)-f(y_{k-1})}{x_k-y_{k-1}}$ (better secant approach).
(III) $\bar{f}'(\alpha) = \frac{f(x_k)-f(z_{k-1})}{x_k-z_{k-1}}$ (best secant approach).
(IV) $\bar{f}'(\alpha) = N_2'(x_k)$ (Newton's interpolatory approach), where $N_2(t) = N_2(t; x_k, z_{k-1}, y_{k-1})$ is Newton's interpolatory polynomial of second degree, set through three best available approximations (nodes) x_k, z_{k-1} and y_{k-1}.

The main idea in constructing methods with memory consists of the calculation of the parameter $\gamma = \gamma_k$ as the iteration proceeds by the formula $\gamma_k = -1/\bar{f}'(\alpha)$ for $k = 1, 2, \ldots$. It is assumed that the initial estimate γ_0 should be chosen before starting the iterative process, for example, using one of the ways proposed in [16, p. 186]. Regarding the above methods (I)–(IV), we present the following four formulas:

$$\gamma_k = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad (\text{Method(I)}), \tag{22}$$
$$\gamma_k = -\frac{x_k - y_{k-1}}{f(x_k) - f(y_{k-1})} \quad (\text{Method(II)}), \tag{23}$$
$$\gamma_k = -\frac{x_k - z_{k-1}}{f(x_k) - f(z_{k-1})} \quad (\text{Method(III)}), \tag{24}$$
$$\gamma_k = -\frac{1}{N_2'(x_k)} \quad (\text{Method(IV)}), \tag{25}$$

where

$$\begin{aligned} N_2'(x_k) &= \left[\frac{d}{dt}N_2(t)\right]_{t=x_k} = \left[\frac{d}{dt}\big(f(x_k) + f[x_k,z_{k-1}](t-x_k) + f[x_k,z_{k-1},y_{k-1}](t-x_k)(t-z_{k-1})\big)\right]_{t=x_k} \\ &= f[x_k,z_{k-1}] + f[x_k,z_{k-1},y_{k-1}](x_k - z_{k-1}) = f[x_k,y_{k-1}] + f[x_k,z_{k-1}] - f[z_{k-1},y_{k-1}]. \end{aligned} \tag{26}$$

Since γ_k is recursively calculated as the iteration proceeds using (I)–(IV), the function φ given by (2) should be replaced by

$$\widetilde{\varphi}(x_k) = \frac{f(x_k + \gamma_k f(x_k)) - f(x_k)}{\gamma_k f(x_k)}. \tag{27}$$

Substituting $\widetilde{\varphi}(x_k)$ instead of φ in (11), we state the following derivative free family of three-point methods with memory,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{\widetilde{\varphi}(x_k)}, \\ z_k = y_k - h(u_k, v_k)\frac{f(y_k)}{\widetilde{\varphi}(x_k)}, \quad (k = 0, 1, \ldots), \\ x_{k+1} = z_k - \frac{f(z_k)}{f[z_k,y_k] + f[z_k,y_k,x_k](z_k-y_k) + f[z_k,y_k,x_k,w_k](z_k-y_k)(z_k-x_k)}, \end{cases} \tag{28}$$

where $\widetilde{\varphi}$ is defined by (27), $w_k = x_k + \gamma_k f(x_k)$, and h is a two-valued weight function that satisfies (4). We use the term *method with memory* following Traub's classification [16, p. 8] and the fact that the evaluation of the parameter γ_k depends on the data available from the current and the previous iterative step. Accelerated methods obtained by recursively calculated free parameter may also be called *self-accelerating methods*.

5. Convergence theorem

To estimate the convergence speed of the family of three-point methods with memory (28), where γ_k is calculated using one of the formulas (22)–(25), we will use the concept of the R-order of convergence introduced by Ortega and Rheinboldt [10]. In our analysis the following assertion is needed (see [1, p. 287]).

Theorem 2. *Let (IM) be an iterative method with memory which generates a sequence $\{x_k\}$ that converges to the zero α, and let $\varepsilon_j = x_j - \alpha$. If there exists a nonzero constant η and nonnegative numbers m_i, $0 \leqslant i \leqslant n$, such that the inequality*

$$|\varepsilon_{k+1}| \leqslant \eta \prod_{i=0}^{n} |\varepsilon_{k-i}|^{m_i}, \quad k \geqslant k(\{\varepsilon_k\}),$$

holds, then the R-order of convergence of iterative method (IM), denoted with $O_R(IM, \alpha)$, satisfies the inequality

$$O_R((IM), \alpha) \geqslant s^*,$$

where s^* is the unique positive zero of the equation

$$s^{n+1} - \sum_{i=0}^{n} m_i s^{n-i} = 0. \tag{29}$$

The proofs of the convergence theorems are given with rigor. However, rigor in itself is not the main object of our analysis and we simplify our proofs omitting those cumbersome details which are of marginal importance and do not influence the final result. For example, to avoid higher order terms in some relations, which make only "parasite" parts of these relations/developments and do not influence the convergence order, we employ the notation used in Traub's book [16]: If $\{f_k\}$ and $\{g_k\}$ are zero-sequences and

$$\frac{f_k}{g_k} \to C,$$

where C is a nonzero constant, we shall write

$$f_k = O(g_k) \quad \text{or} \quad f_k \sim C(g_k).$$

Now we state the convergence theorem for the family (28) of three-point methods with memory.

Theorem 3. *Let the varying parameter γ_k in the iterative scheme (28) be recursively calculated by expressions given in (22)–(25). If an initial approximation x_0 is sufficiently close to the zero α of f, then the R-order of convergence of the three-point methods (28)–(22), (28)–(23), (28)–(24) and (28)–(25) with memory is at least $2(2+\sqrt{5}), 9, 10$ and 11, respectively.*

Proof. Let $\{x_k\}$ be a sequence of approximations generated by an iterative method (IM). If this sequence converges to the zero α of f with the R-order $O_R((IM), \alpha) \geqslant r$, we will write

$$\varepsilon_{k+1} \sim D_{k,r}\varepsilon_k^r, \quad \varepsilon_k = x_k - \alpha, \tag{30}$$

where $D_{k,r}$ tends to the asymptotic error constant D_r of (IM) when $k \to \infty$. Hence

$$\varepsilon_{k+1} \sim D_{k,r}\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^r = D_{k,r}D_{k-1,r}^r\varepsilon_{k-1}^{r^2}. \tag{31}$$

According to the error relations (14), (5) and (21) with the self-accelerating parameter $\gamma = \gamma_k$, we can write the corresponding error relations for the methods (28) with memory

$$\varepsilon_{k,y} = y_k - \alpha \sim c_2(1 + \gamma_k f'(\alpha))\varepsilon_k^2, \tag{32}$$

$$\varepsilon_{k,z} = z_k - \alpha \sim a_{k,4}(1 + \gamma_k f'(\alpha))^2\varepsilon_k^4, \tag{33}$$

$$\varepsilon_{k+1} = x_{k+1} - \alpha \sim a_{k,8}(1 + \gamma_k f'(\alpha))^4\varepsilon_k^8. \tag{34}$$

The expressions of $a_{k,4}$ and $a_{k,8}$ are evident from (5) and (21) and depend on the iteration index since γ_k is recalculated in each iteration. As mentioned above, we omitted higher order terms in (32)–(34).

Let $\varepsilon = x - \alpha$. Using Taylor's series about the root α, we obtain

$$f(x) = f'(\alpha)(\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + O(\varepsilon^5)). \tag{35}$$

This relation will be used for different values of x. Now we determine the R-order of convergence of the family (28) for all approaches (22)–(25) applied to the calculation of γ_k.

Method(I), γ_k is calculated by (22):

Using the development (35) for $x = x_k$ and $x = x_{k-1}$, we obtain

$$\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}} = f'(\alpha)\frac{(\varepsilon_k + c_2\varepsilon_k^2 + c_3\varepsilon_k^3 + \cdots) - (\varepsilon_{k-1} + c_2\varepsilon_{k-1}^2 + c_3\varepsilon_{k-1}^3 + \cdots)}{\varepsilon_k - \varepsilon_{k-1}}$$
$$= f'(\alpha)\left(1 + c_2(\varepsilon_k + \varepsilon_{k-1}) + c_3\left(\varepsilon_k^2 + \varepsilon_k\varepsilon_{k-1} + \varepsilon_{k-1}^2\right) + \cdots\right) = f'(\alpha)\left(1 + c_2(\varepsilon_k + \varepsilon_{k-1}) + O\left(\varepsilon_{k-1}^2\right)\right).$$

According to this, we calculate γ_k by (22) and find

$$1 + \gamma_k f'(\alpha) = c_2(\varepsilon_k + \varepsilon_{k-1}) + O(\varepsilon_{k-1}^2) \sim c_2\varepsilon_{k-1}. \tag{36}$$

Substituting (36) in (34) yields

$$\varepsilon_{k+1} \sim a_{k,8}c_2^4\varepsilon_{k-1}^4\varepsilon_k^8. \tag{37}$$

Hence we can find a constant η so that the inequality

$$|\varepsilon_{k+1}| \leqslant \eta|\varepsilon_k|^4|\varepsilon_{k-1}|^4 \tag{38}$$

holds. Starting from (38) and having in mind Theorem 2 and (29), we form the quadratic equation $r^2 - 8r - 4 = 0$. The positive root $r^* = 2(2+\sqrt{5}) \approx 8.47$ of this equation determines the lower bound of the R-order of convergence of the method (28)–(22).

Method (II), γ_k is calculated by (23):

Similar to the derivation of (36), we calculate γ_k by the more accurate secant method (23) and obtain

$$1 + \gamma_k f'(\alpha) = c_2(\varepsilon_k + \varepsilon_{k-1,y}) + O(\varepsilon_{k-1,y}^2) \sim c_2\varepsilon_{k-1,y}. \tag{39}$$

Assume that the iterative sequence $\{y_k\}$ has the R-order p, then, bearing in mind (30),

$$\varepsilon_{k,y} \sim D_{k,p}\varepsilon_k^p \sim D_{k,p}\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^p = D_{k,p}D_{k-1,r}^p\varepsilon_{k-1}^{rp}. \tag{40}$$

Combining (30), (32), (39) and (40), we get

$$\varepsilon_{k,y} \sim c_2(1+\gamma_k f'(\alpha))\varepsilon_k^2 \sim c_2(c_2\varepsilon_{k-1,y})\varepsilon_k^2 \sim c_2^2\left(D_{k-1,p}\varepsilon_{k-1}^p\right)\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^2 \sim c_2^2 D_{k-1,p}D_{k-1,r}^2\varepsilon_{k-1}^{2r+p}. \tag{41}$$

According to (30), (37) and (40), we obtain

$$\varepsilon_{k+1} \sim a_{k,8}c_2^4\varepsilon_{k-1,y}^4\varepsilon_k^8 \sim a_{k,8}c_2^4\left(D_{k-1,p}\varepsilon_{k-1,y}^p\right)^4\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^8 \sim a_{k,8}c_2^4 D_{k-1,p}^4 D_{k-1,r}^8\varepsilon_{k-1}^{8r+4p}. \tag{42}$$

By comparing exponents of ε_{k-1} on the right-hand side of (40) and (41), and then on the right-hand side of (31) and (42), we form the following system of equations

$$\begin{cases} rp - 2r - p = 0, \\ r^2 - 8r - 4p = 0, \end{cases}$$

with non-trivial solution $p = 9/4$ and $r = 9$. Therefore, the R-order of the methods with memory (28)–(23) is at least *nine*.

Method (III), γ_k is calculated by (24):

Considering the most accurate secant method (24), assume that the iterative sequence $\{z_k\}$ has the R-order s, that is,

$$\varepsilon_{k,z} \sim D_{k,s}\varepsilon_k^s \sim D_{k,s}\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^s \sim D_{k,s}D_{k-1,r}^s\varepsilon_{k-1}^{rs}. \tag{43}$$

Proceeding in the similar way as for the Methods (I) and (II), we start from (24) and obtain

$$1 + \gamma_k f'(\alpha) = c_2(\varepsilon_k + \varepsilon_{k-1,z}) + O\left(\varepsilon_{k-1,z}^2\right) \sim c_2\varepsilon_{k-1,z},$$

which leads to the error relations

$$\varepsilon_{k,z} \sim a_{k,4}(1+\gamma_k f'(\alpha))^2\varepsilon_k^4 \sim a_{k,4}c_2D_{k-1,s}^2D_{k-1,r}^4\varepsilon_{k-1}^{4r+2s} \tag{44}$$

and

$$\varepsilon_{k+1} \sim a_{k,8}(1+\gamma_k f'(\alpha))^4\varepsilon_k^8 \sim a_{k,8}c_2^4D_{k-1,s}^4D_{k-1,r}^8\varepsilon_{k-1}^{8r+4s}. \tag{45}$$

By comparing exponents of ε_{k-1} appearing in two pairs of relations (43)–(44) and (31)–(45),we arrive at the system of equations

$$\begin{cases} rs - 4r - 2s = 0, \\ r^2 - 8r - 4s = 0. \end{cases}$$

Since non-trivial solution of this system is given by $s = 5$ and $r = 10$, we conclude that the R-order of the methods with memory (28-III) is at least *ten*.

Method (*IV*), γ_k *is calculated by* (25):
In view of (26) and (35) we have

$$\begin{aligned}
N_2'(x_k) &= f[x_k, y_{k-1}] + f[x_k, z_{k-1}] - f[z_{k-1}, y_{k-1}] \\
&= \frac{f(x_k) - f(y_{k-1})}{x_k - y_{k-1}} + \frac{f(x_k) - f(z_{k-1})}{x_k - z_{k-1}} - \frac{f(z_{k-1}) - f(y_{k-1})}{z_{k-1} - y_{k-1}} \\
&= \frac{f(x_k) - f(y_{k-1})}{\varepsilon_k - \varepsilon_{k-1,y}} + \frac{f(x_k) - f(z_{k-1})}{\varepsilon_k - \varepsilon_{k-1,z}} - \frac{f(z_{k-1}) - f(y_{k-1})}{\varepsilon_{k-1,z} - \varepsilon_{k-1,y}} \\
&= f'(\alpha)\Bigg[\frac{\varepsilon_k - \varepsilon_{k-1,y} + c_2\left(\varepsilon_k^2 - \varepsilon_{k-1,y}^2\right) + c_3\left(\varepsilon_k^3 - \varepsilon_{k-1,y}^3\right) + \cdots}{\varepsilon_k - \varepsilon_{k-1,y}} + \frac{\varepsilon_k - \varepsilon_{k-1,z} + c_2\left(\varepsilon_k^2 - \varepsilon_{k-1,z}^2\right) + c_3\left(\varepsilon_k^3 - \varepsilon_{k-1,z}^3\right) + \cdots}{\varepsilon_k - \varepsilon_{k-1,z}} \\
&\quad - \frac{\varepsilon_{k-1,z} - \varepsilon_{k-1,y} + c_2\left(\varepsilon_{k-1,z}^2 - \varepsilon_{k-1,y}^2\right) + c_3\left(\varepsilon_{k-1,z}^3 - \varepsilon_{k-1,y}^3\right) + \cdots}{\varepsilon_{k-1,z} - \varepsilon_{k-1,y}}\Bigg] \\
&= f'(\alpha)\left(1 + 2c_2\varepsilon_k + c_3\left(\varepsilon_{k-1,y}\varepsilon_{k-1,z} + \varepsilon_k\varepsilon_{k-1,y} + \varepsilon_k\varepsilon_{k-1,z} + 2\varepsilon_k^2\right) + \cdots\right) = f'(\alpha)(1 + c_3\varepsilon_{k-1,y}\varepsilon_{k-1,z} + O(\varepsilon_k)),
\end{aligned}$$

According to this and (25) we find

$$1 + \gamma_k f'(\alpha) \sim c_3\varepsilon_{k-1,y}\varepsilon_{k-1,z}. \tag{46}$$

Using (46) and the previously derived relations, we obtain the error relations for the intermediate approximations

$$\begin{aligned}
\varepsilon_{k,y} &\sim c_2(1 + \gamma_k f'(\alpha))\varepsilon_k^2 \sim c_2c_3\varepsilon_{k-1,y}\varepsilon_{k-1,z}\varepsilon_k^2 \sim c_2c_3\left(D_{k-1,p}\varepsilon_{k-1}^p\right)\left(D_{k-1,s}\varepsilon_{k-1}^s\right)\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^2 \\
&\sim c_2c_3D_{k-1,p}D_{k-1,s}D_{k-1,r}^2\varepsilon_{k-1}^{2r+s+p},
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\varepsilon_{k,z} &\sim a_{k,4}(1 + \gamma_k f'(\alpha))^2\varepsilon_k^4 \sim a_{k,4}(c_3\varepsilon_{k-1,y}\varepsilon_{k-1,z})^2\varepsilon_k^4 \sim a_{k,4}c_3^2\left(D_{k-1,p}\varepsilon_{k-1}^p\right)^2\left(D_{k-1,s}\varepsilon_{k-1}^s\right)^2\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^4 \\
&\sim a_{k,4}c_3^2D_{k-1,p}^2D_{k-1,s}^2D_{k-1,r}^4\varepsilon_{k-1}^{4r+2s+2p}.
\end{aligned} \tag{48}$$

In the similar fashion we find the error relation for the final approximation within the considered iteration

$$\begin{aligned}
\varepsilon_{k+1} &\sim a_{k,8}(1 + \gamma_k f'(\alpha))^4\varepsilon_k^8 \sim a_{k,8}(c_3\varepsilon_{k-1,y}\varepsilon_{k-1,z})^4\varepsilon_k^8 \sim a_{k,8}c_3^4\left(D_{k-1,p}\varepsilon_{k-1}^p\right)^4\left(D_{k-1,s}\varepsilon_{k-1}^s\right)^4\left(D_{k-1,r}\varepsilon_{k-1}^r\right)^8 \\
&\sim a_{k,8}c_3^4D_{k-1,p}^4D_{k-1,s}^4D_{k-1,r}^8\varepsilon_{k-1}^{8r+4s+4p}.
\end{aligned} \tag{49}$$

Comparing the error exponents of ε_{k-1} in three pairs od relations (40)–(47), (43)–(48), (32)–(49), we form the system of three equations in p, s and r

$$\begin{cases}
rp - 2r - (p + s) = 0, \\
rs - 4r - 2(p + s) = 0, \\
r^2 - 8r - 4(p + s) = 0.
\end{cases}$$

Non-trivial solution of this system is $p = 11/4$, $s = 11/2$, $r = 11$ and we conclude that the lower bound of the R-order of the methods with memory (28)–(25) is *eleven*.

In this way we have completed the analysis of all accelerating methods (22)–(25) so that the proof of Theorem 3 is completed. □

6. Numerical examples

We have tested the family of three-point methods (11) using the programming package *Mathematica* with multiple-precision arithmetic. Apart from this family, several three-point iterative methods (IM) of optimal order eight presented in [2–9,14,17,18], which also require four function evaluations, have been tested. For demonstration, we have selected four methods displayed below.

Three-point methods of Bi et al. [2]:

$$\begin{cases}
y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k = y_k - g(u_k)\frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} = z_k - \frac{f(x_k) + \beta f(z_k)}{f(x_k) + (\beta - 2)f(z_k)} \cdot \frac{f(z_k)}{f[z_k, y_k] + f[z_k, x_k, x_k](z_k - y_k)},
\end{cases} \tag{50}$$

where $\beta \in \mathbf{R}$, $u_k = f(y_k)/f(x_k)$ and $g(u)$ is a real-valued function satisfying

$$g(0) = 1, \quad g'(0) = 2, \quad g''(0) = 10, \quad |g'''(0)| < \infty$$

.

Derivative free Kung–Traub's family [8]:

$$\begin{cases} y_k = x_k - \frac{\gamma f(x_k)^2}{f(x_k+\gamma f(x_k))-f(x_k)}, \\ z_k = y_k - \frac{f(y_k)f(x_k+\gamma f(x_k))}{[f(x_k+\gamma f(x_k))-f(y_k)]f[x_k,y_k]}, \quad (\gamma \in \boldsymbol{R}, k = 0, 1, \ldots), \\ x_{k+1} = z_k - \frac{f(y_k)f(x_k+\gamma f(x_k))\left(y_k - x_k + \frac{f(x_k)}{f[x_k,z_k]}\right)}{[f(y_k)-f(z_k)][f(x_k+\gamma f(x_k))-f(z_k)]} + \frac{f(y_k)}{f[y_k,z_k]}. \end{cases} \tag{51}$$

Kung–Traub's family with first derivative [8]:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(x_k)f(y_k)}{[f(x_k)-f(y_k)]^2}\frac{f(x_k)}{f'(x_k)}, \quad (k = 0, 1, \ldots), \\ x_{k+1} = z_k - \frac{f(x_k)f(y_k)f(z_k)\{f(x_k)^2+f(y_k)[f(y_k)-f(z_k)]\}}{[f(x_k)-f(y_k)]^2[f(x_k)-f(z_k)]^2[f(y_k)-f(z_k)]}\frac{f(x_k)}{f'(x_k)}. \end{cases} \tag{52}$$

Sharma–Sharma's method [14]:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)}{f(x_k)-2f(y_k)}, \\ x_{k+1} = z_k - \left(1 + \frac{f(z_k)}{f(x_k)}\right)\frac{f(z_k)f[x_k,y_k]}{f[x_k,z_k]f[y_k,z_k]}. \end{cases} \tag{53}$$

The errors $|x_k - \alpha|$ of approximations to the zeros, produced by (11), (50)–(52) and (53), are given in Tables 1 and 2, where $A(-h)$ denotes $A \times 10^{-h}$. These tables include the values of the computational order of convergence r_c calculated by the formula [11]

$$r_c = \frac{\log|f(x_k)/f(x_{k-1})|}{\log|f(x_{k-1})/f(x_{k-2})|}, \tag{54}$$

taking into consideration the last three approximations in the iterative process. We have chosen the following test functions:

$$f(x) = e^{x^2+x\cos x-1}\sin \pi x + x\log(x\sin x + 1), \quad \alpha = 0, \quad x_0 = 0.6,$$
$$f(x) = \log(1 + x^2) + e^{x^2-3x}\sin x, \quad \alpha = 0, \quad x_0 = 0.35.$$
$$f(x) = e^{x^2+x\cos x-1}\sin \pi x + x\log(x\sin x + 1), \quad \alpha = 0, \quad x_0 = 0.6, \quad \gamma = -0.1$$
$$f(x) = \log(x^2 - 2x + 2) + e^{x^2-5x+4}\sin(x - 1), \quad \alpha = 1, \quad x_0 = 1.35, \quad \gamma = -0.1$$

From Tables 1 and 2 and many tested examples we can conclude that all implemented methods converge very fast. Although three-point methods from the family (11) produce the best approximations in the case of considered functions, we cannot claim that, in general, they are better than other three-point methods of optimal order eight; numerous tests show that the considered methods generate results of approximately same accuracy. From the last column of Tables 1 and 2 we can also conclude that the computational order of convergence r_c, calculated by (54), matches very well the theoretical order.

The next numerical experiments were performed applying the family (28) of three-point methods with memory to the same functions as above, with the same initial data (x_0 and γ_0). Absolute values $|x_k - \alpha|$ are displayed in Tables 3 and 4. Comparing results given in Tables 3 and 4 (methods with memory) with the corresponding results presented in Tables 1 and 2 (methods without memory), we observe considerable increase of accuracy of approximations produced by the methods with

Table 1
Three-point methods without memory.

Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c (54)
(11) $h(u, v) = (1 + u)/(1 - v)$, $\gamma = -0.1$	0.649(−4)	0.497(−33)	0.586(−266)	8.000
(11) $h(u, v) = 1 + u + v + v^2$	0.645(−4)	0.127(−32)	0.290(−262)	8.000
(11) $h(u, v) = 1 + u + v + (u + v)^2$	0.658(−4)	0.421(−34)	0.117(−275)	7.999
(11) $h(u, v) = u + 1/(1 - v)$	0.645(−4)	0.127(−32)	0.284(−262)	8.000
(50), $g(u) = 1 + \frac{4u}{2-5u}$	0.166(−2)	0.221(−21)	0.221(−172)	7.999
(50), $g(u) = 1 + 2u + 5u^2 + u^3$	0.241(−2)	0.221(−19)	0.118(−155)	7.998
(51), $\gamma = 0.01$	0.126(−2)	0.370(−23)	0.198(−187)	8.000
(52)	0.114(−2)	0.152(−23)	0.154(−190)	8.000
(53)	0.136(−2)	0.279(−23)	0.876(−189)	7.999

Table 2
Three-point methods without memory.

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c (54)
(11) $h(u,v)=(1+u)/(1-v)$	0.288(−5)	0.156(−41)	0.117(−331)	8.000
(11) $h(u,v)=1+u+v+v^2$	0.479(−5)	0.208(−39)	0.262(−314)	8.000
(11) $h(u,v)=1+u+v+(u+v)^2$	0.272(−5)	0.504(−43)	0.701(−345)	7.999
(11) $h(u,v)=u+1/(1-v)$	0.499(−5)	0.291(−39)	0.385(−313)	8.000
(50), $g(u)=1+\frac{4u}{2-5u}$	0.570(−4)	0.898(−31)	0.341(−245)	7.999
(50), $g(u)=1+2u+5u^2+u^3$	0.622(−4)	0.106(−29)	0.772(−236)	7.999
(51), $\gamma=0.01$	0.877(−4)	0.218(−30)	0.314(−243)	7.999
(52)	0.845(−4)	0.169(−30)	0.426(−244)	7.999
(53)	0.782(−4)	0.832(−31)	0.136(−246)	7.999

Table 3
Families of three-point methods with memory.

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c (54)
$h(u,v)=(1+u)/(1-v)$				
(28), (22)	0.649(−4)	0.264(−35)	0.161(−301)	8.481
(28), (23)	0.649(−4)	0.117(−39)	0.460(−359)	8.936
(28), (24)	0.649(−4)	0.177(−41)	0.222(−416)	9.980
(28), (25)	0.649(−4)	0.150(−47)	0.433(−525)	10.944
$h(u,v)=1/[(1-u)(1-v)]$				
(28), (22)	0.653(−4)	0.111(−35)	0.157(−304)	8.462
(28), (23)	0.653(−4)	0.140(−39)	0.208(−358)	8.939
(28), (24)	0.653(−4)	0.192(−41)	0.468(−416)	9.981
(28), (25)	0.653(−4)	0.157(−47)	0.680(−525)	10.944
$h(u,v)=1+u+v+v^2$				
(28), (22)	0.645(−4)	0.108(−34)	0.965(−296)	8.482
(28), (23)	0.645(−4)	0.943(−39)	0.615(−351)	8.962
(28), (24)	0.645(−4)	0.136(−40)	0.199(−407)	10.002
(28), (25)	0.645(−4)	0.138(−46)	0.198(−515)	10.987
$h(u,v)=1+u+v+(u+v)^2$				
(28), (22)	0.658(−4)	0.596(−36)	0.585(−307)	8.458
(28), (23)	0.658(−4)	0.759(−40)	0.833(−361)	8.931
(28), (24)	0.658(−4)	0.103(−41)	0.455(−421)	10.035
(28), (25)	0.658(−4)	0.103(−47)	0.275(−528)	10.971
$h(u,v)=u+1/(1-v)$				
(28), (22)	0.645(−4)	0.108(−34)	0.944(−296)	8.482
(28), (23)	0.645(−4)	0.939(−39)	0.588(−351)	8.962
(28), (24)	0.645(−4)	0.135(−40)	0.182(−407)	10.002
(28), (25)	0.645(−4)	0.110(−46)	0.240(−516)	10.982

memory. The quality of the approaches in calculating γ_k by (22)–(25) can also be observed from Tables 3 and 4: Newton's interpolation gives the best results, which was expected since it provides the highest order 11. The better approximation (among $x_{k-1}, y_{k-1}, z_{k-1}$) is applied in the secant approach (I), (II) or (III), the faster method is obtained. The computational order of convergence, given in the last column of Tables 3 and 4, is not so close to the theoretical value of order as in the case of methods without memory (see Tables 1 and 2), but it is still quite acceptable as a measure of convergence speed having in mind that methods with memory have more complex structure dealing with information from two successive iterations.

The *R*-order of convergence of the family (28) with memory is increased from 8 to $2(2+\sqrt{5})\approx 8.472$, 9, 10 and 11, in accordance with the quality of applied accelerating method given by (22)–(24) or (25). The increase of convergence order is attained without any additional function calculations, which points to a very high computational efficiency of the proposed methods with memory. Finally, note that the order of methods (28) with memory is higher than eight, but it does not refute the Kung–Traub conjecture because this hypothesis is related only to the methods *without memory* such as (11).

Remark 3. From Tables 3 and 4 we notice that approximations produced by (28) using the weight functions $h(u,v)=1+u+v+v^2$ and $h(u,v)=u+1/(1-v)$ are very close to each other. This similarity becomes clearer by observing that $u+\frac{1}{1-v}=(1+u+v+v^2)+v^3+\cdots$.

$$f(x)=e^{x^2+x\cos x-1}\sin\pi x+x\log(x\sin x+1),\quad \alpha=0,\quad x_0=0.6,\quad \gamma_0=-0.1$$

$$f(x)=\log(x^2-2x+2)+e^{x^2-5x+4}\sin(x-1),\quad \alpha=1,\quad x_0=1.35,\quad \gamma_0=-0.1$$

Table 4
Families of three-point methods with memory.

Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c (54)
$h(u,v) = (1+u)/(1-v)$				
(28), (22)	0.288(−5)	0.481(−44)	0.433(−373)	8.486
(28), (23)	0.288(−5)	0.240(−47)	0.621(−426)	8.997
(28), (24)	0.288(−5)	0.135(−49)	0.181(−496)	10.081
(28), (25)	0.288(−5)	0.150(−54)	0.489(−600)	11.069
$h(u,v) = 1/[(1-u)(1-v)]$				
(28), (22)	0.922(−6)	0.172(−47)	0.119(−402)	8.511
(28), (23)	0.922(−6)	0.243(−51)	0.744(−462)	9.006
(28), (24)	0.922(−6)	0.175(−53)	0.255(−535)	10.097
(28), (25)	0.922(−6)	0.194(−58)	0.836(−643)	11.094
$h(u,v) = 1+u+v+v^2$				
(28), (22)	0.479(−5)	0.237(−41)	0.469(−350)	8.503
(28), (23)	0.479(−5)	0.539(−45)	0.944(−405)	9.006
(28), (24)	0.479(−5)	0.277(−47)	0.242(−472)	10.064
(28), (25)	0.479(−5)	0.293(−52)	0.180(−574)	11.061
$h(u,v) = 1+u+v+(u+v)^2$				
(28), (22)	0.272(−5)	0.184(−44)	0.294(−377)	8.496
(28), (23)	0.272(−5)	0.260(−48)	0.138(−434)	8.979
(28), (24)	0.272(−5)	0.234(−50)	0.157(−504)	10.078
(28), (25)	0.272(−5)	0.268(−55)	0.473(−608)	11.054
$h(u,v) = u+1/(1-v)$				
(28), (22)	0.499(−5)	0.332(−41)	0.815(−349)	8.503
(28), (23)	0.499(−5)	0.754(−45)	0.194(−403)	9.005
(28), (24)	0.499(−5)	0.381(−47)	0.580(−471)	10.063
(28), (25)	0.499(−5)	0.407(−52)	0.673(−573)	11.060

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