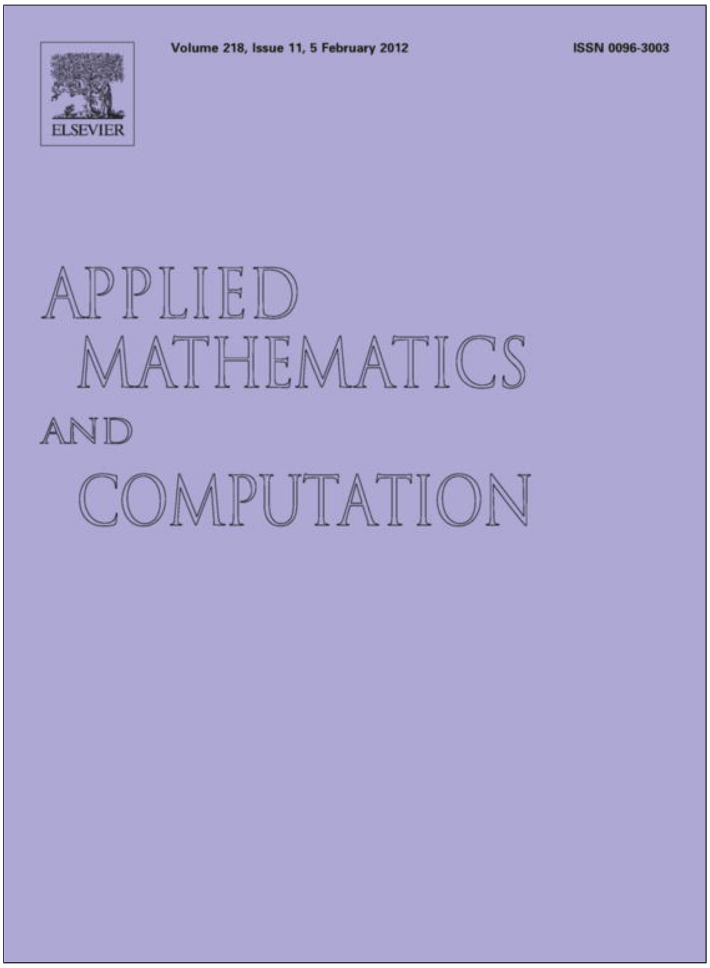
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On optimal fourth-order iterative methods free from second derivative and their dynamics

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ABSTRACT

In this paper new fourth order optimal root-finding methods for solving nonlinear equations are proposed. The classical Jarratt's family of fourth-order methods are obtained as special cases. We then present results which describe the conjugacy classes and dynamics of the presented optimal method for complex polynomials of degree two and three. The basins of attraction of existing optimal methods and our method are presented and compared to illustrate their performance.

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1. Introduction

In this paper, we consider iterative methods and their dynamics to find a simple root ρ, i.e., $f(\rho) = 0$ and $f'(\rho) \neq 0$, of a nonlinear equation $f(x) = 0$. Newton's method [1] is the best known method for finding a real or complex root ρ of the nonlinear equation $f(x) = 0$, which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method converges quadratically in some neighborhood of ρ.

It is also well-known (see [2]) that for any function H with $H(0) = 1$, $H'(0) = 1/2$ and $|H''(0)| < \infty$, the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H(t(x_n)), \tag{1}$$

where

$$t(x_n) = \frac{f(x_n) f''(x_n)}{[f'(x_n)]^2} \tag{2}$$

is of order 3 [2]. The Schemes (1) and (2) include many well-known methods as particular cases; for example, when $H(t) = (1 - \frac{1}{2}t)^{-1}$, $H(t) = 2\left(1 + \sqrt{1-2t}\right)^{-1}$, and $H(t) = (1 - t)^{-0.5}$ it reduces to Halley's method [2], Euler's formula [2], and Ostrowski's square root iteration [3], respectively. The methods (1) and (2) require $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ per step, but it is of third order, so it is not an optimal method. By an optimal method we mean a multipoint one without memory which requires $n + 1$ functional evaluations per iteration, but achieves the order of convergence 2^n [4]. It should be observed that

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the method (1) and (2) even involves the computation of the second derivative of f per step, which restricts its practical use. Optimal root-finding methods which overcome the lack of optimality and practical utility that (1) and (2) has are thus preferred. A new approximation to the second derivative with arbitrarily given second-order method is devised and applied to (1) and (2). The methods derived in this manner will be of order 4 and require one function- and two first derivative-evaluations per step, so they are optimal methods. The methods are also free of second derivative and contain the classical Jarratt's fourth-order methods. Our method developed here also contains Kou et al.'s fourth-order family of methods free from second derivative proposed in [5]. Thus our work can be viewed as an extension of the results of Kou et al. [5]. Sharma and Goyal [6] have developed two fourth-order one-parameter family of methods requiring no evaluation of derivatives.

There are various criteria involved in choosing an iterative method to approximate the root of an equation [7]. These include the initial value problem (for what initial values will the method converge? Will it converge to a root, and if so, which root?), the rate of convergence (how fast the convergence occurs near a root?) and the complexity of the calculation (do first or higher derivatives have to be calculated?). Some of these problems were investigated in [7] by showing how complex dynamics can shed light on them when using Newton's method for finding the real or complex roots of polynomial. In order to investigate these dynamics with some higher order methods we will improve the method (1) and (2) in order to increase the rate of convergence and reduce the complexity of the calculation, and then study their complex dynamics. The dynamics of the König iteration methods [8], the super-Newton method, Cauchy's method, and Halley's methods [9] and a number of root-finding methods including Jarratt's and King's methods [10] were previously studied in detail. Scott et al. compared the dynamics of several methods for simple roots [11] and Neta et al. has performed similar comparison of methods for multiple roots [12]. See also Amat et al. [13]. Motivated by these works this paper thus may be considered as an extension of them in various aspects.

A precise analysis of convergence is given for the presented optimal methods. We present results which describe the conjugacy classes and dynamics of one of the new optimal fourth order methods for complex polynomials of degree two and three. The fact that our method is not generally convergent for polynomials is also investigated by constructing a specific polynomial such that the rational map arising from our method applied to the polynomial has an attracting periodic orbit of period 2. The basins of attraction of some existing fourth order optimal methods and our method are considered and presented. To this end, we shall recall some preliminaries, see for example Milnor [14] and Plaza [10]. Let $R:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational map on the Riemann sphere.

Definition 1. For $z\in\widehat{\mathbb{C}}$ we define its orbit as the set

$$orb(z)=\left\{z,R(z),R^2(z),\ldots,R^n(z),\ldots\right\}.$$

Definition 2. A point z_0 is a fixed point of R if $R(z_0)=z_0$.

Definition 3. A periodic point z_0 of period m is such that $R^m(z_0)=z_0$ where m is the smallest such integer. The set of the m distinct points $\{z,R(z),R^2(z),\ldots,R^{m-1}(z)\}$ is called a periodic cycle.

Remark 1.1. If z_0 is periodic of period m then it is a fixed point for R^m.

Definition 4. If z_0 is a periodic point of period m, then the derivative $(R^m)'(z_0)$ is called the eigenvalue of the periodic point z_0.

Remark 1.2. By the chain rule, if z_0 is a periodic point of period m, then its eigenvalue is the product of the derivatives of R at each point on the orbit of z_0, and we have

$$(R^m)'(z_0)=(R^m)'(z_1)=\cdots=(R^m)'(z_{n-1}),$$

that is, all the points of a cycle have the same eigenvalue.

We classify the fixed points of a map based on the magnitude of the derivative.

Definition 5. A point z_0 is called attracting if $|R'(z_0)|<1$, repelling if $|R'(z_0)|>1$, and neutral if $|R'(z_0)|=1$. If the derivative is zero then the point is called super-attracting.

Definition 6. The Julia set of a nonlinear map $R(z)$, denoted $J(R)$, is the closure of the set of its repelling periodic points. The complement of $J(R)$ is the Fatou set $\mathbb{F}(R)$.

By its definition, $J(R)$ is a closed subset of $\widehat{\mathbb{C}}$. A point z_0 belongs to the Julia set if and only if dynamics in a neighborhood of z_0 displays sensitive dependence on the initial conditions, so that nearby initial conditions lead to wildly different behavior after a number of iterations. As a simple example, consider the map $R(z)=z^2$ on $\widehat{\mathbb{C}}$. The entire open disk is contained in $\mathbb{F}(R)$, since successive iterates on any compact subset converge uniformly to zero. Similarly the exterior is contained in $\mathbb{F}(R)$. On the other hand if z_0 is on the unit circle then in any neighborhood of z_0 any limit of the iterates would necessarily have a jump discontinuity as we cross the unit circle. Therefore $J(R)$ is the unit circle. Such smooth Julia sets are exceptional.

Lemma 1.1 (Invariance Lemma Milnor [14]). *The Julia set J(R) of a holomorphic map $R:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is fully invariant under R. That is, z belongs to J if and only if R(z) belongs to J.*

Lemma 1.2. Iteration Lemma*For any k > 0, the Julia set $J(R^k)$ of the k-fold iterate coincides with J(R).*

Definition 7. If O is an attracting periodic orbit of period m, we define the basin of attraction to be the open set $A\in\widehat{\mathbb{C}}$ consisting of all points $z\in\widehat{\mathbb{C}}$ for which the successive iterates $R^m(z)$, $R^{2m}(z),\ldots$ converge towards some point of O.

The basin of attraction of a periodic orbit may have infinitely many components.

Definition 8. The immediate basin of attraction of a periodic orbit is the connected component containing the periodic orbit.

Lemma 1.3. *Every attracting periodic orbit is contained in the Fatou set of R. In fact the entire basin of attraction A for an attracting periodic orbit is contained in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

2. New iterative methods

Throughout this work let ϕ be an iteration function of order at least two. We let $y_n = x_n - \theta[x_n - \phi(x_n)] = (1-\theta)x_n + \theta\phi(x_n)$, where θ is a nonzero real parameter. Let us consider the approximation:

$$f''(x_n) \approx \frac{f'(y_n)-f'(x_n)}{y_n-x_n} = \frac{f'(x_n)-f'(y_n)}{\theta[x_n-\phi(x_n)]}$$

from which (2) can be approximated

$$t(x_n) = \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2} \approx \frac{f(x_n)[f'(x_n)-f'(y_n)]}{\theta[x_n-\phi(x_n)][f'(x_n)]^2}.$$

This gives rise to a new iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}H(\tilde{t}(x_n)), \tag{3}$$

where

$$\tilde{t}(x_n) = \frac{f(x_n)[f'(x_n)-f'(y_n)]}{\theta[x_n-\phi(x_n)][f'(x_n)]^2}. \tag{4}$$

We will show that in spite of not using as many function evaluations, we have increased the order of convergence to 4. The following theorem will prove that the method defined by (3) and (4) is of order 4 under additional conditions on H and on θ.

Theorem 2.1. *Let $\rho\in I$ be a simple zero of a sufficiently differentiable function $f: I\to\mathbf{R}$ in an open interval I. Let H be any function with $H(0) = 1$, $H'(0) = 1/2$ and $|H''(0)| < \infty$, θ a nonzero real number and ϕ any iteration function of order at least two. Let $y_n = x_n - \theta[x_n - \phi(x_n)]$. Then the method defined by (3) and (4) has third-order convergence, and its error equation is given as*

$$\begin{aligned} e_{n+1} &= \left[2(1-H''(0))c_2^2 + \left(\frac{3}{2}\theta - 1\right)c_3\right]e_n^3 \\ &\quad + \left[\left(14H''(0) - \frac{4}{3}H'''(0) - 9\right)c_2^3 + (6\theta H''(0) - 12H''(0) - 6\theta + 12)c_2c_3 - \frac{3}{2}\theta\phi_2c_3 - (2\theta^2 - 6\theta + 3)c_4\right]e_n^4 + O(e_n^5), \end{aligned} \tag{5}$$

where $e_n = x_n - \rho$,

$$c_k = (1/k!)f^{(k)}(\rho)/f'(\rho), \quad k = 1,2,\ldots \tag{6}$$

$c_0 = f(\rho) = 0$, and $\phi(x_n) = \rho + \phi_2e_n^2 + O(e_n^3)$. Furthermore, if we have $H''(0) = 1$ and $\theta = \frac{2}{3}$, then the order of the method defined by (3) and (4) is at least four.

Proof. Let $e_n = x_n - \rho$ and $d_n = y_n - \rho$, where $y_n = x_n - \theta\psi(x_n)$. Using Taylor expansion and taking into account $f(\rho) = 0$, we have

$$f(x_n)=f'(\rho)\left[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+O(e_n^5)\right], \quad (7)$$

$$f'(x_n)=f'(\rho)\left[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+O(e_n^4)\right] \quad (8)$$

and

$$[f'(x_n)]^2=[f'(\rho)]^2\left[1+4c_2e_n+(4c_2^2+6c_3)e_n^2+12c_2c_3e_n^3+O(e_n^4)\right], \quad (9)$$

where c_k is given by (6).

Dividing (7) by (8) gives

$$\frac{f(x_n)}{f'(x_n)}=e_n-c_2e_n^2+2(c_2^2-c_3)e_n^3+(7c_2c_3-3c_4-4c_2^3)e_n^4+O(e_n^5). \quad (10)$$

Since ϕ is an iteration function of order at least two, it follows that

$$\phi(x_n)=\rho+\phi_2e_n^2+\phi_3e_n^3+\phi_4e_n^4+O(e_n^5),$$

where $\phi_k=\frac{1}{k!}\phi^{(k)}(\rho),\ k=2,3,4$ so that

$$x_n-\phi(x_n)=e_n-\phi_2e_n^2-\phi_3e_n^3-\phi_4e_n^4+O(e_n^5) \quad (11)$$

and hence, we have

$$d_n=e_n-\theta[x_n-\phi(x_n)]=(1-\theta)e_n+\theta\phi_2e_n^2+\theta\phi_3e_n^3+\theta\phi_4e_n^4+O(e_n^5). \quad (12)$$

Expanding $f'(y_n)$ about ρ, we have

$$f'(y_n)=f'(\rho)\left[1+2c_2d_n+3c_3d_n^2+4c_4d_n^3+5c_5d_n^4+O(d_n^5)\right]$$

and then from (12), we obtain

$$\begin{aligned}f'(y_n)=f'(\rho)\Big[&1+2(1-\theta)c_2e_n+\left[2\theta\phi_2c_2+3(1-\theta)^2c_3\right]e_n^2+\left[2\theta\phi_3c_2+6\theta(1-\theta)\phi_2c_3+4(1-\theta)^3c_4\right]e_n^3\\&+\left[2\theta\phi_4c_2+3\theta(\theta\phi_2^2+2(1-\theta)\phi_3)c_3+12\theta(1-\theta)^2\phi_2c_4+5(1-\theta)^4c_5\right]e_n^4+O(e_n^5)\Big].\end{aligned}$$

It is then clear that

$$\begin{aligned}f'(x_n)-f'(y_n)=f'(\rho)\Big[&2\theta c_2e_n+[3c_3-2\theta\phi_2c_2-3(1-\theta)^2c_3]e_n^2+[4c_4-2\theta\phi_3c_2-6\theta(1-\theta)\phi_2c_3-4(1-\theta)^3c_4]e_n^3\\&-[2\theta\phi_4c_2+3\theta(\theta\phi_2^2+2(1-\theta)\phi_3)c_3+12\theta(1-\theta)^2\phi_2c_4+5(1-\theta)^4c_5-5c_5]e_n^4+O(e_n^5)\Big].\end{aligned} \quad (13)$$

By a simple calculation, we have from (7), (9), (11) and (13) that

$$\begin{aligned}\tilde{t}(x_n)&=\frac{f(x_n)[f'(x_n)-f'(y_n)]}{\theta[x_n-\phi(x_n)][f'(x_n)]^2}\\&=2c_2e_n+\left[3(2-\theta)c_3-6c_2^2\right]e_n^2+\left[16c_2^3+(9\theta-28)c_2c_3+3\theta\phi_2c_3+4(\theta^2-3\theta+3)c_4\right]e_n^3+O(e_n^4)\end{aligned} \quad (14)$$

and so,

$$\tilde{t}^2(x_n)=4c_2^2e_n^2+4c_2\left[6c_3-3\theta c_3-6c_2^2\right]e_n^3+O(e_n^4). \quad (15)$$

From (14) and (15), we have upon using the values of $H(0)$ and $H'(0)$

$$\begin{aligned}H(\tilde{t}(x_n))&=1+\frac{1}{2}\tilde{t}(x_n)+\frac{1}{2}H''(0)\tilde{t}^2(x_n)+\frac{1}{6}H'''(0)\tilde{t}^3(x_n)+O(\tilde{t}^4(x_n))\\&=1+c_2e_n+\left[(2H''(0)-3)c_2^2+\frac{3}{2}(2-\theta)c_3\right]e_n^2+\left[\left(8-12H''(0)+\frac{4}{3}H'''(0)\right)c_2^3\right.\\&\quad\left.+\left(12H''(0)-6\theta H''(0)+\frac{9}{2}\theta-14\right)c_2c_3+\frac{3}{2}\theta\phi_2c_3+2(\theta^2-3\theta+3)c_4\right]e_n^3+O(e_n^4).\end{aligned} \quad (16)$$

Hence, from (10) and (16), we obtain

$$\begin{aligned}x_{n+1}&=x_n-\frac{f(x_n)}{f'(x_n)}H(\tilde{t}(x_n))\\&=\rho+\left[2(1-H''(0))c_2^2+\left(\frac{3}{2}\theta-1\right)c_3\right]e_n^3\\&\quad+\left[\left(14H''(0)-\frac{4}{3}H'''(0)-9\right)c_2^3+(6\theta H''(0)-12H''(0)-6\theta+12)c_2c_3-\frac{3}{2}\theta\phi_2c_3-(2\theta^2-6\theta+3)c_4\right]e_n^4+O(e_n^5),\end{aligned}$$

therefore,

$$e_{n+1} = \left[2(1-H''(0))c_2^2 + \left(\frac{3}{2}\theta - 1\right)c_3\right]e_n^3 + \left[\left(14H''(0) - \frac{4}{3}H'''(0) - 9\right)c_2^3 + (6\theta H''(0) - 12H''(0) - 6\theta + 12)c_2c_3 - \frac{3}{2}\theta\phi_2c_3 - (2\theta^2 - 6\theta + 3)c_4\right]e_n^4 + O(e_n^5),$$

which is the same one that appears in (5).

Now if we choose $\theta = \frac{2}{3}$ and $H''(0) = 1$, then (2) becomes

$$e_{n+1} = \left[\left(14H''(0) - \frac{4}{3}H'''(0) - 9\right)c_2^3 + (6\theta H''(0) - 12H''(0) - 6\theta + 12)c_2c_3 - \frac{3}{2}\theta\phi_2c_3 - (2\theta^2 - 6\theta + 3)c_4\right]e_n^4 + O(e_n^5) \tag{17}$$

and we obtain the fourth-order class of methods

$$y_n = x_n - \frac{2}{3}[x_n - \phi(x_n)], \tag{18}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}H(\tilde{t}(x_n)), \tag{19}$$

where

$$\tilde{t}(x_n) = \frac{3}{2}\frac{f(x_n)[f'(x_n) - f'(y_n)]}{[x_n - \phi(x_n)][f'(x_n)]^2} \tag{20}$$

and ϕ is any iteration function of order at least two. This completes the proof. □

If we consider an iterative function ϕ requiring $f(x_n)$ and $f'(x_n)$, then our family of methods has an optimal order since it requires $f(x_n)$, $f'(x_n)$ and $f'(y_n)$ per step.

3. New fourth order optimal methods

For the sake of simplicity, we consider only Newton's iteration function $\phi(x) = x - \frac{f(x)}{f'(x)}$, even though other choices for ϕ may provide us with many other optimal fourth-order methods. For the Newton iteration function, (18)–(20) simplifies to

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \tag{21}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}H(\tilde{t}(x_n)), \tag{22}$$

where

$$\tilde{t}(x_n) = \frac{3}{2}\frac{f'(x_n) - f'(y_n)}{f'(x_n)}. \tag{23}$$

If we take $H(t) = 1 + \frac{1}{2}\frac{t}{1-t}$, then (21)–(23) leads to the well-known Jarratt's fourth-order method [15]

$$x_{n+1} = x_n - \left[1 - \frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)},$$

where $y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

If we take another $H(t) = 1 + \frac{9}{6-4t} - \frac{9}{6-2t}$, then (21)–(23) leads to another optimal fourth-order Jarratt's method [15]

$$x_{n+1} = x_n - w_1(x_n) - \frac{3}{2}w_2(x_n) + \frac{3f(x_n)}{f'(x_n) + f'(z_n)},$$

where $w_1(x_n) = \frac{f(x_n)}{f'(x_n)}$, $w_2(x_n) = \frac{f(x_n)}{f'(z_n)}$ and $z_n = x_n - \frac{2}{3}w_1(x_n)$. This method is suggested by Jarratt in order to reduce the possibility of cancelation in the denominator.

If we take $H(t) = \frac{3}{4}\frac{t(\gamma t + \frac{3}{2})}{(\alpha t + \frac{3}{2})(\beta t + \frac{3}{2})}$, where $\gamma = \alpha + \beta - \frac{3}{2}, \alpha, \beta \in \mathbf{R}$, then (21)–(23) leads to the optimal Kou et al.'s fourth-order family of methods [5]

$$x_{n+1} = x_n - \left(1 - \frac{3}{4}\frac{(f'(y_n) - f'(x_n))(\gamma f'(y_n) + (1-\gamma)f'(x_n))}{(\alpha f'(y_n) + (1-\alpha)f'(x_n))(\beta f'(y_n) + (1-\beta)f'(x_n))}\right)\frac{f(x_n)}{f'(x_n)}.$$

In the case that $H(t)=1+\frac{t}{2}+\frac{t^2}{2}$, (21)–(23) gives a new fourth-order optimal method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left[1+\frac{3}{4}\frac{f'(x_n)-f'(y_n)}{f'(x_n)}+\frac{9}{8}\left(\frac{f'(x_n)-f'(y_n)}{f'(x_n)}\right)^2\right],$$

where $y_n=x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

In the case that $H(t)=1+\frac{2}{t-2}+\frac{4}{(t-2)^2}$, we obtain from (21)–(23) another new optimal fourth-order method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left[1-\frac{4f'(x_n)}{3f'(y_n)+f'(x_n)}+\left(\frac{4f'(x_n)}{3f'(y_n)+f'(x_n)}\right)^2\right],$$

where $y_n=x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

In the case that $H(t)=-\frac{t}{2}-\frac{4}{t-2}-1$, (21)–(23) reduces to the method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left[\frac{3}{4}\frac{f'(y_n)-f'(x_n)}{f'(x_n)}+\frac{8f'(x_n)}{3f'(y_n)+f'(x_n)}-1\right],$$

where $y_n=x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

In the case that $H(t)=\frac{4}{4-2t-t^2}$, we obtain from (21)–(23) another new optimal fourth-order method

$$x_{n+1}=x_n-\frac{16f(x_n)f'(x_n)}{-5[f'(x_n)]^2+30f'(x_n)f'(y_n)-9[f'(y_n)]^2}, \tag{24}$$

where $y_n=x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

4. Conjugacy classes

Throughout the remainder of this paper we study the dynamics of the rational map R_f arising from the method (24)

$$R_f(z)=z+\frac{16f(z)f'(z)}{5[f'(z)]^2-30f'(z)f'(y)+9[f'(y)]^2}, \tag{25}$$

where

$$y=z-\frac{2}{3}\frac{f(z)}{f'(z)}$$

applied to a generic polynomial with simple roots. We tried other possibilities and they are not competitive. Let us first recall the definition of analytic conjugacy classes.

Definition 9 [16]. Let f and g be two maps from the Riemann sphere into itself. An analytic conjugacy between f and g is an analytic diffeomorphism h from the Riemann sphere onto itself such that $h\circ f=g\circ h$.

R_f has the following useful property for an analytic function f.

Theorem 4.1 (*The Scaling Theorem*). *Let $f(z)$ be an analytic function on the Riemann sphere, and let $T(z)=\alpha z+\beta,\alpha\neq 0$, be an affine map. If $g(z)=f\circ T(z)$, then $T\circ R_g\circ T^{-1}(z)=R_f(z)$. That is, R_f is analytically conjugate to R_g by T.*

Proof. With the iteration function $R(z)$, we have

$$R_g(T^{-1}(z))=T^{-1}(z)+16g'(T^{-1}(z))g(T^{-1}(z))\left[5g'^2(T^{-1}(z))-30g'(T^{-1}(z))g'\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)\right.$$
$$\left.+9g'^2\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)\right]^{-1}.$$

Since $g\circ T^{-1}(z)=f(z)$, $(g\circ T^{-1})'(z)=\frac{1}{\alpha}g'(T^{-1}(z))$, we get $g'(T^{-1}(z))=\alpha\,(g\circ T^{-1})'(z)=\alpha f'(z)$, $g''(T^{-1}(z))=\alpha^2 f''(z)$. We therefore have

$$T\circ R_g\circ T^{-1}(z)=T(R_g(T^{-1}(z)))=\alpha R_g(T^{-1}(z))+\beta=\alpha T^{-1}(z)+\alpha\cdot 16g'(T^{-1}(z))g(T^{-1}(z))$$
$$\times\left[5g'^2(T^{-1}(z))-30g'(T^{-1}(z))g'\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)+9g'^2\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)\right]^{-1}+\beta$$
$$=z+16\alpha^2f'(z)f(z)\left[5\alpha^2[f'(z)]^2-30\alpha f'(z)g'\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)+9g'^2\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)\right]^{-1}. \tag{26}$$

On the other hand, we have

$$g'\left(T^{-1}(z)-\frac{2}{3}\frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right)=g'\left(T^{-1}(z)-\frac{2}{3}\frac{f(z)}{\alpha f'(z)}\right)=g'(T^{-1}(z))-g''(T^{-1}(z))\frac{2}{3}\frac{f(z)}{\alpha f'(z)}+\cdots$$
$$=\alpha f'(z)-\alpha^2 f''(z)\frac{2}{3}\frac{f(z)}{\alpha f'(z)}+\cdots=\alpha\left(f'(z)-f''(z)\frac{2}{3}\frac{f(z)}{f'(z)}+\cdots\right)=\alpha f'\left(z-\frac{2}{3}\frac{f(z)}{f'(z)}\right)=\alpha f'(y).$$

We thus obtain from (26) $T\circ R_g\circ T^{-1}(z)=R_f(z)$, this completing the proof. □

The scaling theorem established above indicates that up to a suitable change of coordinates the study of the dynamics of the iteration function (25) for polynomials can be reduced to the study of the dynamics of the same iteration function for simpler polynomials. For example, for any quadratic and any cubic polynomials, we can easily prove the following results by an affine change of variable and multiplication by a constant that.

Theorem 4.2. *Let $p(z)=az^2+bz+c$, with $a\neq 0$ and*

$$q(z)=z^2-\mu, \tag{27}$$

where $\mu=\frac{b^2-4ac}{4a}$. Then there is an analytic conjugacy between R_p and R_q.

Theorem 4.3. *Let $p(z)=(z-z_0)(z-z_1)(z-z_2)$, with $0\leqslant|z_0|\leqslant|z_1|\leqslant|z_2|$ and let*

$$q(z)=z^3+(\lambda-1)z-\lambda,\quad \lambda\in\mathbb{C}. \tag{28}$$

Then there is an analytic conjugacy between R_p and R_q.

Thus analyzing the iteration function (25) for any quadratic and cubic reduces to analyzing it for the q's in (27) and (28), respectively.

Definition 10 [9]. We say that a one-point iterative root-finding algorithm $p\to T_p$ has a universal Julia set (for polynomials of degree d) if there exists a rational map S such that for every degree d polynomial p, $J(T_p)$ is conjugate by a Möbius transformation to $J(S)$.

The following theorem establishes a universal Julia set for quadratics for our method (24).

Theorem 4.4. *For a rational map $R_p(z)$ arising from the method (24) applied to $p(z)=(z-a)(z-b)$, $a\neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z)=\frac{z-a}{z-b}$ to*

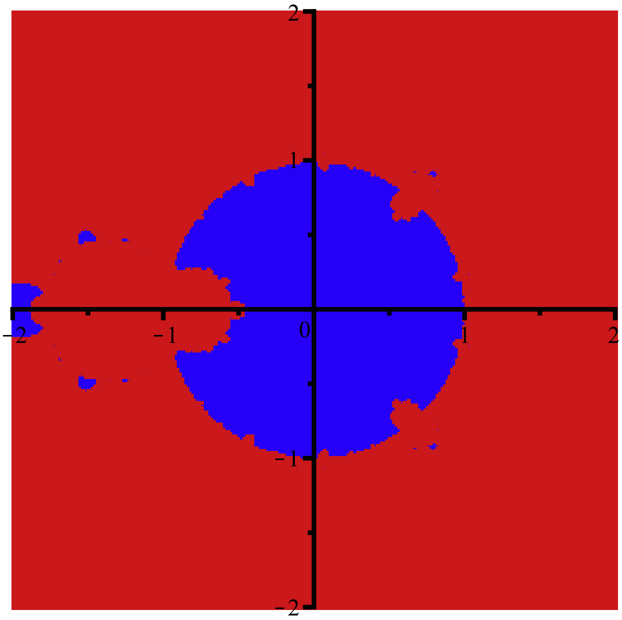

Fig. 1. Basin of attraction for $S(z)=z^4\,\frac{z+2}{2z+1}$.

$$S(z) = z^4 \frac{z+2}{2z+1}$$

Proof. Let $p(z) = (z - a)(z - b)$, $a \neq b$ and Let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u^4\left[\frac{u+2}{2u+1}\right]. \qquad \square$$

Fig. 1 illustrates the dynamic structure of $S(z) = z^4 \frac{z+2}{2z+1}$. The basin of attraction for $S(z)$ clearly reveals the structure of the universal Julia set for S when p is quadratic. The points in the blue area converge to the origin, the red area points converge to the point at infinity.

5. Fixed points and critical points

In the following theorem, we establish the dynamical characterization regarding the fixed points of R_p.

Theorem 5.1. *Assume p is a generic polynomial of degree $d \geqslant 2$ with simple roots. If z_0 is a simple root of p, then it is a super-attracting fixed point of R_p. All other additional fixed points of R_p are roots of $p'(z) = 0$.*

Proof. Let $p(z)$ be a generic polynomial of degree $d \geqslant 2$ with simple roots. Suppose that z_0 is a root of $p(z)$. Then R_p satisfies that $R_p(z_0) = z_0,\ R_p^{(j)}(z_0) = 0,\ j = 1, 2, 3,\ R_p^{(4)}(z_0) \neq 0$ since it is of order four [1]. Hence R_p has a super-attracting fixed point at each root of p. Since $R_p(z_0) = z_0$ only when $p(z_0) = 0$ or $p'(z_0) = 0$, all other additional fixed points of R_p are roots of $p'(z) = 0$. Note that if $p'(z_0) = 0$, then we have $p(z_0) \neq 0$ since z_0 would otherwise be a multiple root. $\square$

For a generic polynomial p, additional fixed points of R_p and their dynamical behavior can be found and determined by Theorem 5.1. For example, for the cubic polynomial $p(z) = z^3 - 1$, R_p has the additional fixed point $z_0 = 0$. Since $|R_p'(0)| = 1$ (see (5)), it is an indifferent fixed point, this altering the basins of attraction of the roots of the cubic.

Critical values of a function f are those values $v \in \mathbb{C}$ for which $f(z) = v$ has a multiple root. The multiple root $z = c$ is called the *critical point* of f. This is equivalent to the condition $f'(c) = 0$. Let $p(z)$ be a generic polynomial with simple roots. The free critical points of R_p are those critical points that are not roots of $p(z)$. The underlying reason for studying the free critical points is due to the following well-known fact.

Theorem 5.2 (*Fatou-Julia*). *Let $R(z)$ be a rational map. If z_0 is an attracting periodic point, then the immediate basin of attraction $B^*(z_0)$ contains at least one critical point.*

As a consequence of Theorem 5.2, it is important to detect the existence of attracting periodic cycles. If there exist an attracting periodic cycle, then there exists at least one critical point near the cycle, and the iterates of R_p starting with the critical point converge to that cycle and not to a root. Thus the existence of attracting periodic cycles could interfere with our R_p search for a root of the equation $p(z) = 0$. To detect the existence of attracting periodic cycles, the orbits of the free critical points of the R_p function should be observed and their set of limit points determined.

Upon differentiating (25) we have

$$\begin{aligned} R_p'(z) = 1 + [5p'^2(z) - 30p'(z)p'(y) + 9p'^2(y)]^{-2}\big[(16p'^2(z) + 16p(z)p''(z))(5p'^2(z) - 30p'(z)p'(y) + 9p'^2(y)) \\ - 16p(z)p'(z)\left(10p'(z)p''(z) - 30p''(z)p'(y) - 30p'(z)p''(y)\frac{p'^2(z) + 2p(z)p''(z)}{3p'^2(z)} + 18p'(y)p''(y)\frac{p'^2(z) + 2p(z)p''(z)}{3p'^2(z)}\right)\bigg]. \quad (29) \end{aligned}$$

It follows from (29) that the equation for the critical points of the iterative method R_p is given by

$$\begin{aligned} 0 = p'(z)(5p'^2(z) - 30p'(z)p'(y) + 9p'^2(y))(21p'^2(z) - 30p'(z)p'(y) + 9p'^2(y) + 16p(z)p''(z)) - 16p(z)[10p'^3(z)p''(z) \\ - 30p'^2(z)p''(z)p'(y) - 2(p'^2(z) + 2p(z)p''(z))(5p'(z)p''(y) - 3p'(y)p''(y))]. \end{aligned}$$

For the cubic $p(z) = z^3 - 1$, we have $R_p = \frac{z(206z^{12} + 544z^9 - 6z^6 - 14z^3 - 1)}{449z^{12} + 301z^9 - 6z^6 - 14z^3 - 1}$ which is of degree $d = 13$. $R_p'(z) = 0$ gives us

$$R_p'(z) = \frac{(92494z^{15} + 36994z^{12} + 2062z^9 - 298z^6 - 31z^3 - 1)(z^3 - 1)^3}{(449z^{12} + 301z^9 - 6z^6 - 14z^3 - 1)^2}.$$

So, the critical points of R_p are roots of the equation $(92494\mu^5 + 36994\mu^4 + 2062\mu^3 - 298\mu^2 - 31\mu - 1)(\mu - 1)^3 = 0$, where $\mu = z^3$. The roots are

$$\begin{aligned} z^3 = 1, 1, 1, -0.301495723260134, -0.110677387489009, 0.102269175468264, -0.0450285716377835 \\ \pm 0.0337719969533516i. \end{aligned}$$

R_p has $d+1=14$ fixed points and $2d-2=24$ critical points in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, counting multiplicity. The use of Maple shows that nine of these critical points z_0 have eigenvalue $|R'(z_0)|=1$, and the rest have $|R'(z_0)|<1$.

Given a polynomial $p(z)$, an iteration function $T_p(z)$ is said to be *generally convergent* if for almost all $z \in \mathbb{C}$ its orbit converges to a root of $p(z)$. The fact that Newton's method is not generally convergent for polynomials was investigated by Barna [17]. We also investigate this aspect for the method (24) by constructing a specific polynomial $p(z)$ such that the rational map R_p arising from (24) applied to the polynomial has an attracting periodic orbit of period 2. Our approach is based on an argument of Smale [18,19] and we have the following result.

Proposition 5.1. *The method (24) is not generally convergent for the polynomial*

$$p(z) = z^3 + az^2 + bz + c,$$

where

$$a = -0.24862826203703351651,$$
$$b = -1.0865745919948851643,$$
$$c = 0.72915055282835616542.$$

Proof. Consider the polynomial $p(z) = z^3 + a\,z^2 + b\,z + c$. We find the coefficients a, b and c so that R_p arising from (24) applied to $p(z)$ will have a super-attracting periodic point of period 2 at the origin, that is, $R_p(0)=1,\ R_p(1)=0,\ R'_p(0)=0,\ R'_p(1)\neq\infty$, which by the chain rule would give $R_p^2(0)=0,\ \left(R_p^2\right)'(0) = R'_p(R_p(0))R'_p(0) = R'_p(1)R'_p(0) = 0$. Hence there exists an open neighborhood of the origin such that the fixed point iteration does not converge to any of the roots of $p(z)$. Therefore the method (24) is not generally convergent for this polynomial. The conditions $R_p(0)=1,\ R_p(1)=0,\ R'_p(0)=0$ imply that a, b and c are the solution of the system

$$c^3(c^5 - 2c^3b^3 + 2cb^6 - 4c^4ab + 16c^2ab^4 - 4ab^7 + 6c^3a^2b^2 - 22ca^2b^5 - 4c^2a^3b^3 + 8a^3b^6 + ca^4b^4)$$
$$\times (b^3c^2 + b^6 - c^4 - b^4ac + 2c^3ab - a^2c^2b^2)^{-2} = 0, \tag{30}$$

$$\frac{16cb}{5b^2 - 30\mu_1 b + 9\mu_1^2} = 1, \tag{31}$$

$$\frac{16(1+a+b+c)(3+2a+b)}{5(3+2a+b)^2 - 30\mu_2(3+2a+b) + 9\mu_2^2} = -1, \tag{32}$$

where

$$\mu_1 = \frac{4c^2}{3b^2} - \frac{4ac}{3b} + b,$$
$$\mu_2 = 3\left[1 - \frac{2(1+a+b+c)}{3(3+2a+b)}\right]^2 + 2a\left[1 - \frac{2(1+a+b+c)}{3(3+2a+b)}\right] + b.r.$$

Solving the system (30)–(32) by Maple with the number of digits set to 20 produces

$$a = -0.24862826203703351651,$$
$$b = -1.0865745919948851643,$$
$$c = 0.72915055282835616542.$$

Thus the polynomial

$$p(z) = z^3 - 0.24862826203703351651z^2 - 1.0865745919948851643z + 0.72915055282835616542$$

makes the method (24) fail to converge to a root of $p(z)$ over a set of positive Lebesgue measure. □

6. Numerical results

Four fourth order optimal methods are considered, which are King's method [20] with $\beta = -\frac{1}{2}$ given by

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)}\frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}, \tag{33}$$

Kung–Traub's method [4] given by

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= \frac{f(w_n)}{f'(x_n)} \frac{1}{[1 - f(w_n)/f(x_n)]^2}, \end{aligned} \quad (34)$$

Kou et al.'s method [21] given by

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f^2(x_n) + f^2(w_n)}{f'(x_n)[f(x_n) - f(w_n)]} \end{aligned} \quad (35)$$

and our method (24).

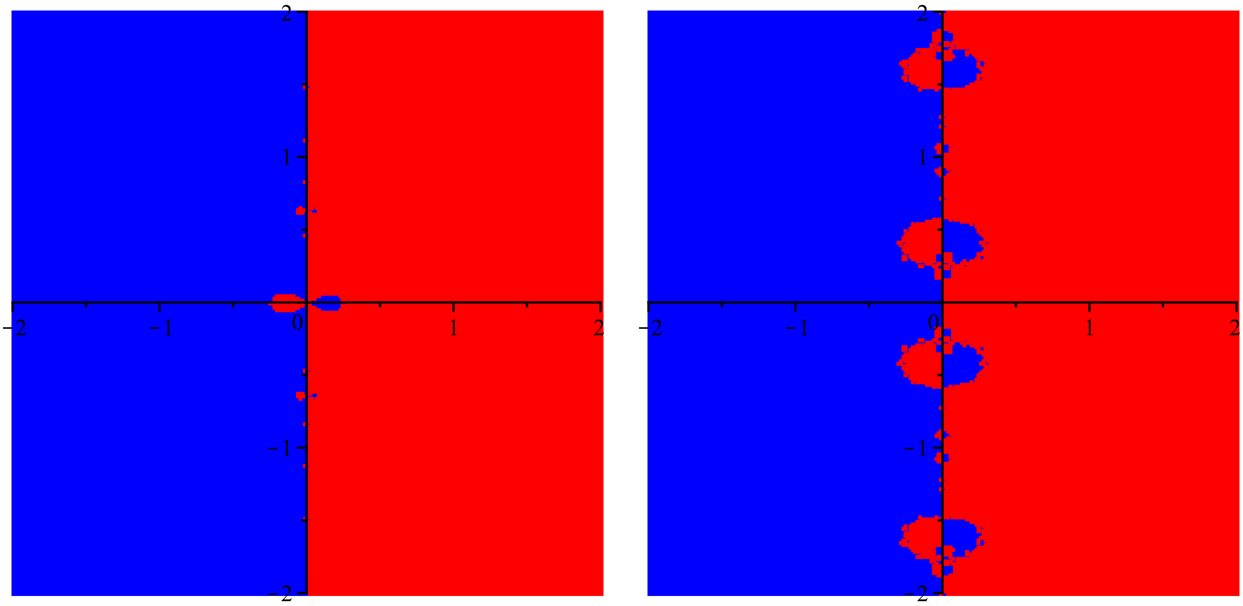

Fig. 2. King's with $\beta = -\frac{1}{2}$ (left) and Kung–Traub's method (right) for the roots of the quadratic polynomial $z^2 - 1$.

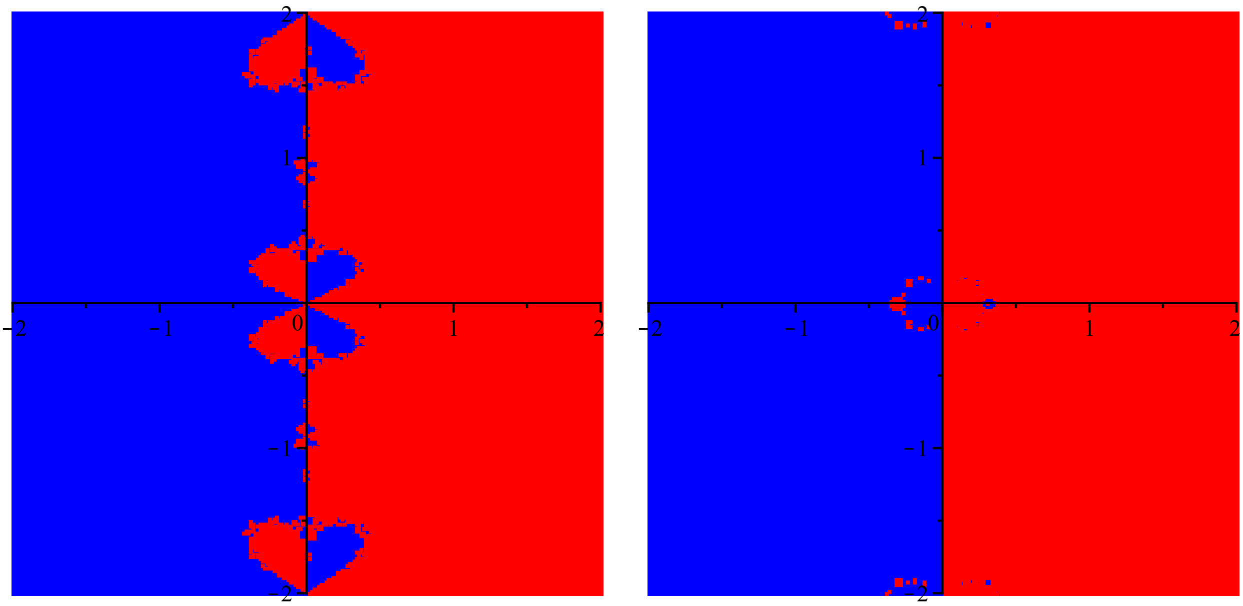

Fig. 3. Kou's method (left) and the method (24) (right) for the roots of the quadratic polynomial $z^2 - 1$.

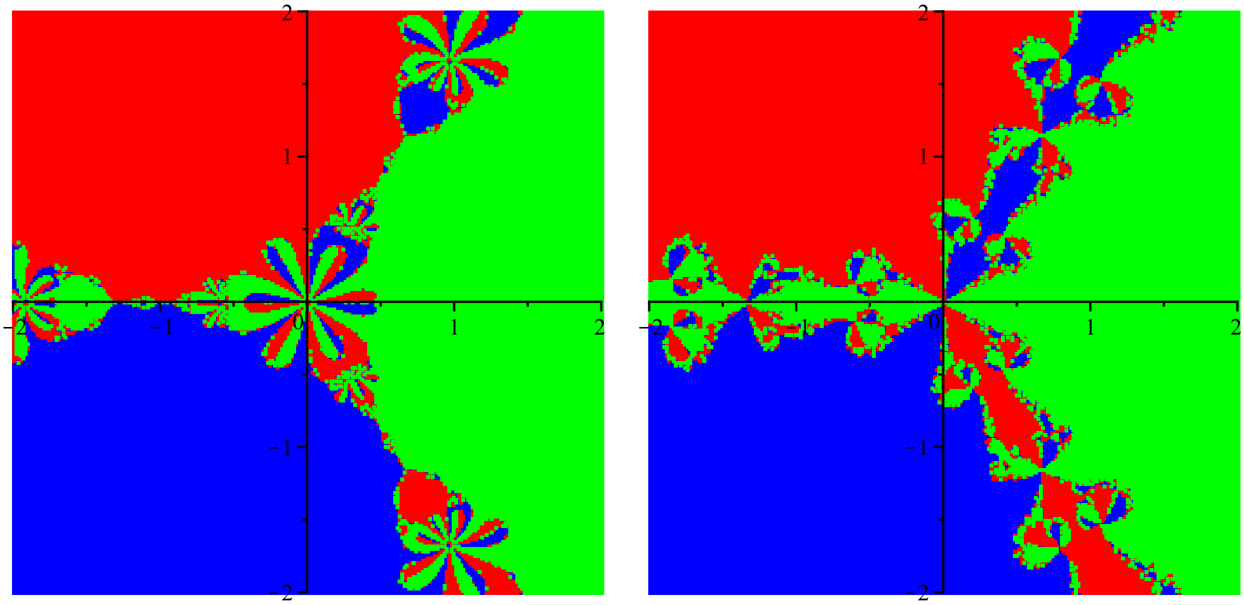

Fig. 4. King's with $\beta = -\frac{1}{2}$ (left) and Kung–Traub's method (right) for the roots of the cubic polynomial $z^3 - 1$.

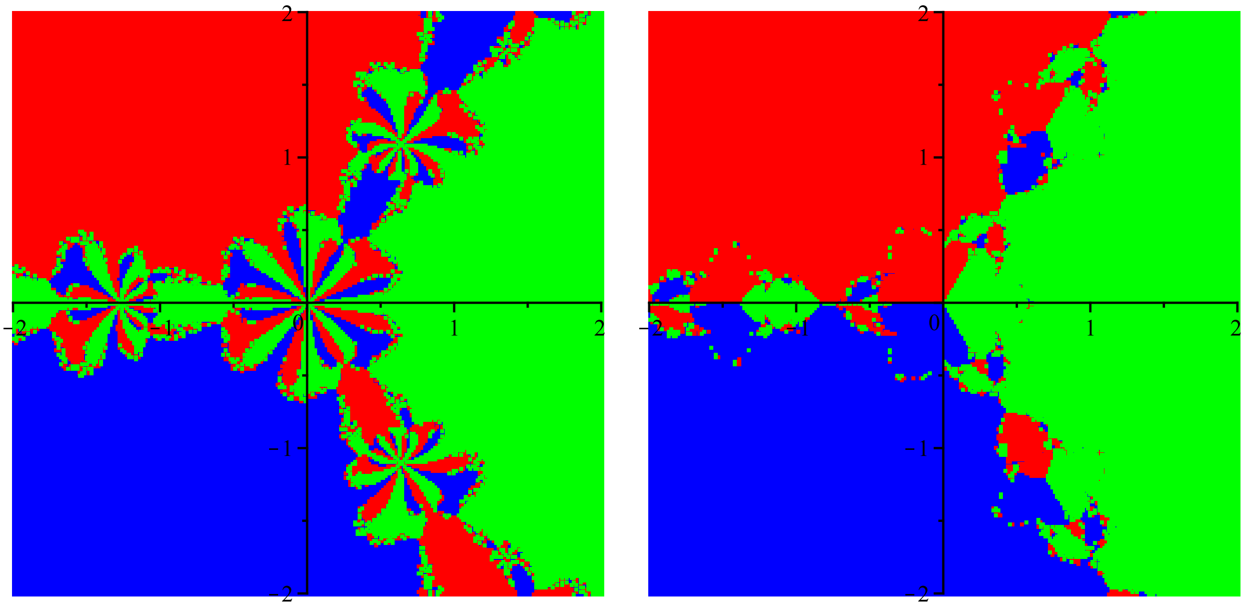

Fig. 5. Kou's method (left) and the method (24) (right) for the roots of the cubic polynomial $z^3 - 1$.

The basins of attraction of the four methods applied to the quadratic polynomial $z^2 - 1$ are presented and compared in Figs. 2 and 3. The results for the cubic polynomial $z^3 - 1$ are given in Figs. 4 and 5.

The basin of attraction for the method (24) is better than any of the other methods. For our method (24) applied to the quadratic polynomial with distinct roots, an almost arbitrary point converges to the root closer to the point.

7. Conclusion

In this paper we have constructed new optimal fourth order root-finding methods for solving nonlinear equations, which contains well-known Jarratt's methods as special cases. We presented results which describe the conjugacy classes and dynamics of presented optimal methods for complex polynomials of degree two and three. We constructed a specific polynomial such that the rational map arising from our method applied to the polynomial has an attracting periodic orbit of period 2. The basins of attraction of existing optimal methods and our method are considered to deal with initial value problems of iteration methods and compared to illustrate their performance as a criterion for comparison.

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