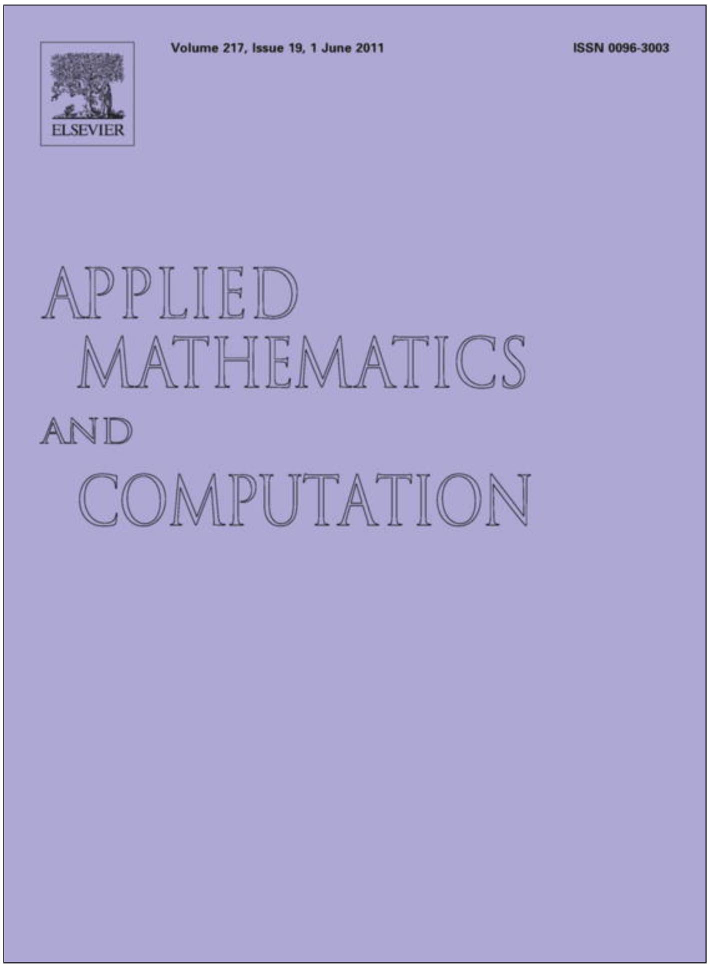
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A family of optimal three-point methods for solving nonlinear equations using two parametric functions

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ABSTRACT

Using an interactive approach which combines symbolic computation and Taylor's series, a wide family of three-point iterative methods for solving nonlinear equations is constructed. These methods use two suitable parametric functions at the second and third step and reach the eighth order of convergence consuming only four function evaluations per iteration. This means that the proposed family supports the Kung–Traub hypothesis (1974) on the upper bound 2^m of the order of multipoint methods based on $m+1$ function evaluations, providing very high computational efficiency. Different methods are obtained by taking specific parametric functions. The presented numerical examples demonstrate exceptional convergence speed with only few function evaluations.

1. Introduction

Multipoint iterative methods for solving nonlinear equations appeared for the first time in Ostrowski's book [1] and then they were extensively studied in Traub's book [2] and some papers published in the 1960s and 1970s. This class of methods has drawn a considerable attention in recent years, which led to the construction of many methods of this type. The reason for the revived interest in this area is a nice property of multipoint methods to overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency, which is of great practical importance. Simply, multipoint methods are primarily introduced with the aim to achieve as high as possible order of convergence using a fixed number of function evaluations. This is closely connected to the *optimal order* of convergence in the sense of the Kung–Traub hypothesis. Namely, studying the optimal convergence rate of multipoint methods, Kung and Traub [3] conjectured that *multipoint methods without memory for solving nonlinear equations, based on* $m+1$ *function evaluations per iteration, have the order of convergence at most* 2^m.

In this paper we develop a new family of very efficient three-point methods for solving nonlinear equations. This family has the order of convergence eight and uses only four function evaluations per iteration. In this way, the optimal order of convergence and optimal computational efficiency in the sense of the Kung–Traub conjecture are attained. The new family was derived in a simple and elegant way and produces a variety of specific methods. The construction and convergence analysis of this family are given in Section 2. The proposed family is tested on numerical examples and compared with the existing three-point methods with optimal order eight (Section 3).

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2. New family of optimal three-point methods

Let α be a simple real root of a real function $f: D \subset \mathbf{R} \to \mathbf{R}$ and let x_0 be an initial approximation to α. We start with a *naive* three-point method:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(y_k)}{f'(y_k)} \quad (k = 0, 1, \ldots), \\ x_{k+1} = \varphi(x_k) = z_k - \frac{f(z_k)}{f'(z_k)}. \end{cases} \tag{1}$$

According to Traub's Theorem 2.4 [2], this method has the order $2 \cdot 2 \cdot 2 = 8$, but it requires six function evaluations per iteration. We define the efficiency index I of a method (M) as $I(M) = r^{1/\theta}$ (see Traub [2]), where r is the order of convergence of the method (M) and θ is the number of function- and derivative-evaluation per iteration. The efficiency index of the three-point method (1) is $I(1) = 8^{1/6} = 2^{1/2}$, which is equal to that of Newton's method. In fact, we do apply Newton's method three times.

To improve the computational efficiency of the iterative method (1), we will modify the above scheme in order to decrease the number of function evaluations per iteration but keeping the order eight. Our approach consists of the elimination of the derivatives $f'(y)$ and $f'(z)$ at the second and third step of (1) using the approximations

$$f'(y) = \frac{f'(x)}{p(s)}, \quad f'(z) = \frac{f'(x)}{q(s,t)}, \quad \text{where} \quad s = \frac{f(y)}{f(x)}, t = \frac{f(z)}{f(y)} \tag{2}$$

and p and q are some functions of one and two variables, respectively. We notice that the quantities s and t do not require new information since they are expressed by the already calculated quantities. The functions p and q are called *multiplicative functions* or *multipliers*.

We start from the scheme (1) and the approximations (2) and state the following family of three-point methods:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - p(s_k)\frac{f(y_k)}{f'(x_k)} \quad (k = 0, 1, \ldots), \\ x_{k+1} := \varphi(x_k) = z_k - q(s_k, t_k)\frac{f(z_k)}{f'(x_k)}. \end{cases} \tag{3}$$

The functions p and q should be determined in such way that the iterative method (3) attains the order eight. To do that, we will use the method of undetermined coefficients and Taylor's series about 0 for $p(s)$ and about $(0,0)$ for $q(s,t)$, thus,

$$p(s) = p(0) + p'(0)s + \frac{p''(0)}{2!}s^2 + \frac{p'''(0)}{3!}s^3 + \cdots, \tag{4}$$

$$\begin{aligned} q(s,t) = {} & q(0,0) + q_s(0,0)s + q_t(0,0)t + \frac{1}{2!}\left[q_{ss}(0,0)s^2 + 2q_{st}(0,0)st + q_{tt}(0,0)t^2\right] \\ & + \frac{1}{3!}\left[q_{sss}(0,0)s^3 + 3q_{sst}(0,0)s^2t + 3q_{stt}(0,0)st^2 + q_{ttt}(0,0)t^3\right] + \cdots \end{aligned} \tag{5}$$

Here the subscripts denote the respective partial derivatives; for example, $q_s(0,0) = \frac{\partial q(s,t)}{\partial s}\Big|_{(s,t)=(0,0)}$, $q_{st}(0,0) = \frac{\partial^2 q(s,t)}{\partial s \partial t}\Big|_{(s,t)=(0,0)}$, etc. Let $e_k = x_k - \alpha$ be the error at the kth iteration. For simplicity, we sometimes omit the iteration index and write e instead of e_k. The expressions of Taylor's polynomials (in e) of the functions involved in (3) are cumbersome and lead to tedious calculations which can be realized only by a computer program. To find the coefficients $p(0), p'(0), p''(0), q(0,0), \ldots, q_{ttt}(0,0)$ of the developments (4) and (5), we have used symbolic computation in the programming package *Mathematica* and an interactive approach explained by the comments C1–C7 given below. We emphasize that all expressions can be easily displayed using the presented program. In this way we avoid writing cumbersome relations and expressions, which are often presented in papers devoted to this topic. Note that these complicated expressions are found by no means using a paper-and-pencil methods but by a computer program.

First, let us introduce the following abbreviations used in the presented program.

`ck` = $f^{(k)}(\alpha)/(k!f'(\alpha))$, `e` = $x - \alpha$, `e1` = $\varphi(x) - \alpha$.
`fx` = $f(x)$, `fy` = $f(y)$, `fz` = $f(z)$, `f1x` = $f'(x)$, `f1a` = $f'(\alpha)$,
`p0` = $p(0)$, `p1` = $p'(0)$, `p2` = $p''(0)$, `p3` = $p'''(0)$,
`q0` = $q(0,0)$, `qs` = $q_s(0,0)$, `qt` = $q_t(0,0)$,
`qss` = $q_{ss}(0,0)$, `qst` = $q_{st}(0,0)$, `qtt` = $q_{tt}(0,0)$,
`qsss` = $q_{sss}(0,0)$, `qsst` = $q_{sst}(0,0)$, `qstt` = $q_{stt}(0,0)$, `qttt` = $q_{ttt}(0,0)$.

Program (written in *Mathematica*)

```
fx=f1a*(e+c2*e^2+c3*e^3+c4*e^4+c5*e^5+c6*e^6+c7*e^7+c8*e^8);
f1x=D[fx,e]; d=Series[fx/f1x,{e,0,8}]; y=x-d; u=e-d;
fy=f1a*(u+c2*u^2+c3*u^3+c4*u^4+c5*u^5+c6*u^6+c7*u^7+c8*u^8);
s=Series[fy/fx,{e,0,8}]; f1y=f1x/(p0+p1*s+p2*s^2/2+p3*s^3/6);
r1=Series[fy/f1y,{e,0,8}]; z=y-r1; v=u-r1;
b2=Coefficient[v,e^2]
```

C1: Out[b2] $= c2$ (1-p0)

p0=1 ; b3=Coefficient[v,e^3]

C2: Out[b3] $= c2^2$ (2-p1)

p1=2

```
fz=f1a*(v+c2*v^2+c3*v^3+c4*v^4+c5*v^5+c6*v^6+c7*v^7+c8*v^8);
t=Series[fz/fy,{e,0,8}];
f1z=f1x/(q0+qs*s+qt*t+qss*s^2/2+qst*s*t+ qtt*t^2/2
+1/6*(qsss*s^3 +3qsst*s^2*t+3*qstt*s*t^2 +qttt*t^3));
r2=Series[fz/f1z,{e,0,8}]; e1=v-r2 // Simplify; a4=Coefficient[e1,e^4]
```

C3: Out[a4] $= \frac{1}{2}c2(2c3 + c2^2(-10 + p2))$ (-1+q0)

q0=1 ; a5=Coefficient[e1,e^5]//Simplify

C4: Out[a5] $= \frac{1}{2}c2^2(2c3 + c2^2(-10 + p2))$ (-2+qs)

qs=2 ; a6=Coefficient[e1,e^6]//Simplify

C5: Out[a6] $= -\frac{1}{4}c2(2c3 + c2^2(-10 + p2))(2c3$ (-1+qt) $+ c2^2$ (12-qss+(-10+p2)qt))

qt=1; qss=2+p2 ; a7=Coefficient[e1,e^7]//Simplify

C6: Out[a7] $= -\frac{1}{12}c2^2(2c3 + c2^2(-10 + p2))(c2^2$ (96+p3-qsss+ 3p2(-2+qst) -30qst)+6c3(-4+qst))

qst=4; p3=qsss; p2=4; qss=6 ; a8=Coefficient[e1,e^8]//Simplify

C7: Out[a8] $= \frac{1}{6}c2(3c2^2 - c3)(6c2c4 - 3c3^2(qtt - 2) + 3c2^2c3(6qtt + qsst - 34) + c2^4(2qsss - 9(3qtt + qsst - 18)))$

From the expression of the error $e_1 = \hat{x} - \alpha = \varphi(x) - \alpha$ we observe that e_1 is of the form

$$e_1 = \varphi(x) - \alpha = a_2e^2 + a_3e^3 + a_4e^4 + a_5e^5 + a_6e^6 + a_7e^7 + a_8e^8 + O(e^9). \qquad (6)$$

The iterative three-point method $x_{k+1} = \varphi(x_k)$ will have the order of convergence equal to eight if we determine the coefficients of the developments appearing in (4) and (5) in such way that $a_2, a_3, a_4, a_5, a_6, a_7$ (in (6)) vanish. We find these coefficients equalling shaded expressions in boxed formulas to 0.

Comment C1: First, to provide the fourth order of convergence of the iterative method consisting of the first two steps of the iterative scheme (3), it is necessary to obtain the error $v = z - \alpha = O(e^4)$. In other words, the coefficients b_2 and b_3 in the expression $v = b_2e^2 + b_3e^3 + b_4e^4$ should vanish. From `Out[b2]` we have $-1 + p0 = 0$ so that we take $p0 = 1$ (given in the shaded box) to eliminate b_2.

Comment C2: From `Out[a3]` we have the equation $2 - p1 = 0$ and we take $p1 = 2$ to vanish b_3.

We continue in the similar way and from **C3–C7** we find the remaining coefficients of the developments (4) and (5):

$$q0 = 1, \quad qs = 2, \quad qt = 1, \quad qs = 2, \quad qst = 4,$$

$$p3 = qsss = \gamma, \quad p2 = 4, \quad qss = 6, \quad qtt, qsst, \gamma \ \text{ arbitrary.}$$

Remark 1. The relation in **C6** does not deliver the unique coefficients. We take $p3 = qsss = \gamma$, where γ is an arbitrary parameter, and derive the expression of a_8 in the form given in **C7**. In practice, the choice $\gamma = 0$ is preferable for simplicity. We do not have to worry about the terms t^2, st^2 and t^3 because they are of higher order and do not influence the order of convergence (not greater than 8).

In this way we have proved the following assertion.

Theorem 1. *If p and q are arbitrary real functions with Taylor's series of the form*

$$p(s) = 1 + 2s + 2s^2 + \gamma s^3 + \cdots, \tag{7}$$

$$q(s,t) = 1 + 2s + t + 3s^2 + 4st + \gamma s^3 + \cdots \quad (\gamma \in \boldsymbol{R}), \tag{8}$$

then the family of three-point methods (3) has the order eight. It is assumed that the terms of higher order in (7) and (8), which follow after the dots, can take arbitrary values.

Remark 2. The entries s_k and t_k in (2) are calculated using the already found quantities $f(x_k)$, $f(y_k)$ and $f(z_k)$ so that the total number of function evaluations per iteration of the method (3) is four. According to this fact and Theorem 1, it follows that the iterative method (3) is *optimal* in the Kung–Traub sense and has the efficiency index $I(3) = 8^{1/4} \approx 1.682$.

The functions p and q in (3) can take many forms satisfying the conditions (7) and (8). In practice, it is reasonable to choose p and q as simple as possible, for example, in the form of a rational function as follows:

$$p(s) = \frac{1 + (2-\beta)s(1+s)}{1 - \beta s(1-s)}, \tag{9}$$

$$q(s,t) = \frac{2 + (6+\gamma_1)s^2 + 2(t+\gamma_2) + s(4 + 2\gamma_1 + \gamma_3 t)}{2 + 2\gamma_1 s - 3\gamma_1 s^2 + \gamma_2 t + (-8 - 2\gamma_1 - 2\gamma_2 + \gamma_3)st}, \tag{10}$$

where β, γ_1, γ_2, γ_3 are arbitrary constants. For example, we give the following specific forms:

$$\begin{aligned} &p_1(s) = 1 + 2s + 2s^2, && p_2(s) = \tfrac{1}{1-2s+2s^2}, && p_3(s) = \tfrac{1+s+s^2}{1-s+s^2}, \\ &q_1(s,t) = 1 + 2s + t + 3s^2 + 4st, && q_2(s,t) = \left(2s + \tfrac{5}{4}t + \tfrac{1}{1+s+\frac{3}{4}t}\right)^2, && \\ &q_3(s,t) = \tfrac{1-4s+t}{(1-3s)^2+2st}, && q_4(s,t) = \tfrac{1}{1-2s+s^2+4s^3-t}. && \end{aligned} \tag{11}$$

All functions except q_2 and q_4 follow from (9) and (10).

Remark 3. According to the comment **C7**, the asymptotical error constant (AEC, for brevity) of the method $x_{k+1} = \varphi(x_k)$, given by (6), is

$$\begin{aligned} AEC(3) = a_8 &= \lim_{k\to\infty} \frac{\varphi(x_k) - \alpha}{(x_k - \alpha)^8} \\ &= \frac{1}{6} c_2(3c_2^2 - c3)\left(6c_2c_4 - 3c_3^2(qtt - 2) + 3c_2^2c_3(6qtt + qsst - 34) + c_2^4(2qsss - 9(3qtt + qsst - 18))\right) \end{aligned}$$

and depends on arbitrary constants $q_{tt}(0,0)$, $q_{sst}(0,0)$ and $q_{sss}(0,0)$. However, this expression is correct if the Taylor series in (4) and (5) are truncated to the displayed terms, that is, for p_1 and q_1 given in the list (11). For a particular pair of functions p and q the asymptotical error constant a_8 is given by a specific expression which can be determined directly by substituting p and q in the iterative scheme (3). For example, choosing two pairs of functions (p_2, q_3) and (p_3, q_4) (listed in (11)) into the iterative formula (3), using a standard convergence technique with the help of symbolic computation and Taylor's series we obtain

$$AEC((3)\text{—}p_2\text{—}q_3) = c_2(3c_2^2 - c_3)(17c_2^4 - 6c_2^2c_3 + c_3^2 + c_2c_4),$$

$$AEC((3)\text{—}p_3\text{—}q_4) = c_2^2(c_2c_3^2 + 3c_2^2c_4 - c_3c_4 - 9c_2^5).$$

3. Numerical results

The family of three-point methods (3) have been tested on a number of nonlinear equations. The programming package *Mathematica* 7 with multi-precision arithmetic (500 significant decimal digits) was employed to provide very high accuracy. The new family (3) has been compared to the existing three-point methods (some of them are given below) which have the same convergence rate (eight) and require four function evaluations per iteration too. We consider that the comparison to the methods having the efficiency index less than $8^{1/4}$ is not necessary since these methods are not competitive. Instead, we compared only methods of the same efficiency index $8^{1/4}$.

3.1. Kung–Traub's method (12) – version without derivatives

Using the inverse interpolatory polynomial of degree $n-1$, Kung and Traub [3] derived two families of multipoint methods of arbitrary order of convergence 2^{n-1}. Taking $n = 4$ the following derivative free three-point method of eighth order is obtained:

$$\begin{cases} y_k = x_k - \frac{\gamma f(x_k)^2}{f(x_k+\gamma f(x_k))-f(x_k)}, \\ z_k = y_k - \frac{f(y_k)f(x_k+\gamma f(x_k))}{(f(x_k+\gamma f(x_k))-f(y_k))f[x_k,y_k]} \quad (k=0,1,\ldots), \\ x_{k+1} = z_k - \frac{f(y_k)f(x_k+\gamma f(x_k))\left(y_k-x_k+\frac{f(x_k)}{f[x_k,z_k]}\right)}{(f(y_k)-f(z_k))(f(x_k+\gamma f(x_k))-f(z_k))} + \frac{f(y_k)}{f[y_k,z_k]}, \end{cases} \tag{12}$$

where γ is a real parameter and $f[x,y] = (f(x) - f(y))/(x - y)$ denotes a divided difference.

3.2. Kung–Traub's method (13) – version with derivative

Taking $n = 4$ in the generalized iterative formula which uses the first derivative [3], the following eight-order method is obtained:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(x_k)^2 f(y_k)}{f'(x_k)(f(x_k)-f(y_k))^2} \quad (k=0,1,\ldots), \\ x_{k+1} = z_k - \frac{f(x_k)^2 f(y_k)}{\Delta_{yz}^{(k)}}\left[\frac{1}{\Delta_{xz}^{(k)}}\left(\frac{x_k-z_k}{\Delta_{xz}^{(k)}} - \frac{1}{f'(x_k)}\right) - \frac{f(y_k)}{f'(x_k)\left(\Delta_{xy}^{(k)}\right)^2}\right], \end{cases} \tag{13}$$

where, for example, $\Delta_{xz}^{(k)} = f(x_k) - f(z_k)$.

3.3. Three-point methods (14) of Bi, Wu and Ren

The eighth-order family of iterative methods proposed by Bi et al. [4] is given in the form:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - h(\mu_k)\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \frac{f(x_k)+\beta f(z_k)}{f(x_k)+(\beta-2)f(z_k)} \cdot \frac{f(z_k)}{f[z_k,y_k]+f[z_k,x_k,x_k](z_k-y_k)} \quad (\beta \in \mathbf{R}), \end{cases} \tag{14}$$

where $\mu_k = f(y_k)/f(x_k)$, $h(t)$ is a suitably chosen real-valued function and $f[z,x,x] = \frac{f[z,x]-f'(x)}{z-x}$. We have tested two methods belonging to the family (14), obtained by choosing two different forms of the function h (see Tables 2–5).

3.4. Three-point methods (15) of Wang and Liu

The eighth-order family of optimal iterative methods proposed by Wang and Liu [5] gives a number of specific methods among which we have selected the following one:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \frac{f(x_k)}{f'(x_k)}\frac{f(x_k)-f(y_k)}{f(x_k)-2f(y_k)}, \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}\left[1 + \frac{4f(z_k)}{f(x_k)+af(z_k)}\right]\left[\frac{f(x_k)^2}{f(x_k)^2-2f(x_k)f(y_k)-f(y_k)^2} + \frac{f(z_k)}{f(y_k)}\right] \quad (a \in \boldsymbol{R}). \end{cases} \tag{15}$$

Note that the first two steps make Ostrowski's two-point method.

3.5. Three-point methods (16) of Neta and Petković

Using inverse interpolation and any optimal two-point method at the first two-steps, a family of three-point method of order eight was presented in [6]. In particular, the use of King's fourth-order method [7] gives

Table 1
Tested functions and initial approximations.

Example k	Function f_k	Root α	Initial approximations x_0
1	$e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	-1	-0.3
2	$x^2 - (1-x)^{25}$	0.1437392592...	0.4
3	$e^{-x} - \arctan 2x - 1$	0	0.5
4	$\prod_{j=1}^{12}(x-j)$	9	9.5

Table 2
$f(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1,\ \alpha = -1,\ x_0 = -0.3.$

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$
Method (3)–p_1–q_1	6.32(−5)	2.97(−37)	7.00(−296)
Method (3)–p_1–q_2	2.64(−5)	2.37(−39)	9.94(−312)
Method (3)–p_1–q_3	2.18(−4)	8.63(−33)	5.27(−260)
Method (3)–p_2–q_1	4.92(−5)	4.61(−38)	2.70(−302)
Method (3)–p_2–q_2	4.39(−5)	1.40(−37)	1.51(−297)
Method (3)–p_2–q_3	2.42(−4)	2.24(−32)	1.18(−256)
Method (3)–p_3–q_1	5.72(−5)	1.43(−37)	2.22(−298)
Method (3)–p_3–q_2	3.39(−5)	1.77(−38)	9.74(−305)
Method (3)–p_3–q_3	2.28(−4)	1.32(−32)	1.71(−258)
Kung–Traub's method (12), $\gamma = 0.02$	1.50(−4)	1.80(−33))	7.59(−265)
Kung–Traub's method (13)	1.11(−4)	9.99(−35)	4.34(−275)
(14), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	1.87(−4)	6.46(−33)	1.30(−260)
(14), $h(t) = (1 - 2t - t^2 + 4t^3)^{-1}$, $\beta = 3$	1.18(−4)	1.35(−34)	3.90(−274)
Wang–Liu's method (15), $a = 0$	7.16(−5)	3.47(−36)	1.06(−286)
Neta–Petkovi'c's method (16), $\lambda = 2$	1.79(−4)	3.50(−33)	7.59(−263)

Table 3
$f(x) = x^2 - (1-x)^{25}$, $\alpha = 0.14373925929975369826\ldots$, $x_0 = 0.4$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$
Method (3)–p_1–q_1	1.12(−3)	1.06(−16)	7.19(−121)
Method (3)–p_1–q_2	5.00(−3)	5.71(−12)	2.48(−83)
Method (3)–p_1–q_3	1.03(−3)	8.21(−18)	1.36(−130)
Method (3)–p_2–q_1	1.70(−3)	1.55(−15)	7.99(−112)
Method (3)–p_2–q_2	4.94(−3)	1.33(−12)	1.41(−89)
Method (3)–p_2–q_3	1.11(−3)	6.16(−17)	5.64(−123)
Method (3)–p_3–q_1	1.36(−3)	3.79(−16)	1.48(−116)
Method (3)–p_3–q_2	4.98(−3)	1.89(−12)	1.92(−87)
Method (3)–p_3–q_3	1.06(−3)	2.72(−17)	5.10(−126)
Kung–Traub's method (12), $\gamma = 0.02$	4.05(−3)	4.07(−14)	2.49(−102)
Kung–Traub's method (13)	3.92(−3)	2.90(−14)	1.54(−103)
(14), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	4.80(−4)	2.59(−19)	1.95(−141)
(14), $h(t) = (1 - 2t - t^2 + 4t^3)^{-1}$, $\beta = 3$	6.57(−3)	1.35(−10)	5.13(−72)
Wang–Liu's method (15), $a = 0$	1.59(−2)	1.52(−9)	3.71(−65)
Neta–Petkovi'c's method (16), $\lambda = 2$	2.25(−3)	1.48(−15)	3.49(−113)

Table 4
$f(x) = e^{-x} - \arctan 2x - 1$, $\alpha = 0$, $x_0 = 0.5$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$
Method (3)–p_1–q_1	4.28(−2)	2.09(−20)	6.24(−159)
Method (3)–p_1–q_2	1.25(−2)	8.42(−19)	1.25(−147)
Method (3)–p_1–q_3	1.36(−3)	1.52(−24)	3.72(−192)
Method (3)–p_2–q_1	2.42(−2)	1.21(−14)	7.83(−113)
Method (3)–p_2–q_2	2.99(−3)	2.72(−23)	1.61(−183)
Method (3) p_2–q_3	3.24(−3)	1.50(−21)	3.38(−168)
Method (3)–p_3–q_1	4.77(−3)	4.14(−20)	1.47(−156)
Method (3)–p_3–q_2	1.09(−2)	1.92(−20)	9.46(−161)
Method (3)–p_3–q_3	4.72(−3)	2.98(−20)	8.04(−158)
Kung–Traub's method (12), $\gamma = 0.02$	2.72(−3)	3.97(−23)	8.71(−182)
Kung–Traub's method (13)	3.33(−3)	2.52(−22)	2.91(−175)
(14), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	0.562	9.49(−6)	2.60(−43)
(14), $h(t) = (1 - 2t - t^2 + 4t^3)^{-1}$, $\beta = 3$	0.217	3.82(−8)	2.48(−62)
Wang–Liu's method (15), $a = 0$	8.87(−3)	4.15(−18)	1.09(−140)
Neta–Petkovi'c's method (16), $\lambda = 2$	1.40(−2)	2.05(−17)	6.19(−136)

$$\begin{cases} w_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = w_k - \frac{f(w_k)}{f'(x_k)} \cdot \frac{f(x_k)+\lambda f(w_k)}{f(x_k)+(\lambda-2)f(w_k)}, \quad (\lambda \in \mathbf{R}), \\ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} + A_k[f(x_k)]^2 - B_k[f(x_k)]^3, \end{cases} \tag{16}$$

where

$$B_k = \frac{1}{\Delta_f^k(w,x)\Delta_f^k(w,z)f[w_k,x_k]} - \frac{1}{\Delta_f^k(z,x)\Delta_f^k(w,z)f[z_k,x_k]} + \frac{1}{f'(x_k)\Delta_f^k(z,x)\Delta_f^k(w,z)} - \frac{1}{f'(x_k)\Delta_f^n(w,x)\Delta_f^k(w,z)}$$

and

$$A_k = \frac{1}{\Delta_f^k(w,x)f[w_k,x_k]} - \frac{1}{f'(x_k)\Delta_f^n(w,x)} - B_k\Delta_f^k(w,x)$$

with the abbreviation $\Delta_f^k(w,z) = f(w_k) - f(z_k)$.

Beside the aforementioned methods (12)–(16), we also tested different methods from the new family (3) choosing various pairs (p_i,q_j) of multiplicative functions given by (11). Among many numerical experiments, we have selected four examples for demonstration presented in Table 1. The absolute errors $|x_k - \alpha|$ in the first three iterations are given in Tables 2–5, where $A(-h)$ means $A \times 10^{-h}$.

Note that several recent papers [8–14] have presented new optimal three-point methods of the order eight. We have found that they give results of approximately the same quality as the presented ones. For this reason and to save space, results of these methods are not included in Tables 2–5.

From the results displayed in Tables 2–5 and a number of numerical experiments, it can be concluded that the proposed multipoint method (3) is competitive with existing three-point methods of optimal order eight and possesses very fast convergence for good initial approximations. All tested variants of (3) demonstrated similar behavior for various pairs (p_i,q_j) of multiplicative functions given by (11). We emphasize a convenient property of this family that it allows the construction of a variety of methods with different forms.

The computational order of convergence, evaluated by the approximate formula

$$\tilde{r} \approx \frac{\log|f(x_{k+1})/f(x_k)|}{\log|f(x_k)/f(x_{k-1})|},$$

Table 5
$f(x) = W_{12}(x) = \prod_{j=1}^{12}(x-j),\ \alpha = 9,\ x_0 = 9.5.$

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$
Method (3)–p_1–q_3	7.69(−3)	5.75(−16)	6.46(−121)
Method (3)–p_2–q_3	5.69(−3)	4.65(−17)	1.02(−129)
Method (3)–p_3–q_3	6.48(−3)	1.40(−16)	7.29(−126)
Kung–Traub's method (12), $\gamma = 0.02$	2.32(−4)	1.78(−16)	5.58(−113)
Kung–Traub's method (13)	0.162	3.67(−7)	4.68(−51)
(14), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	0.158	2.40(−8)	1.59(−60)
(14), $h(t) = (1 - 2t - t^2 + 4t^3)^{-1}$, $\beta = 3$	0.147	3.34(−8)	9.63(−60)
Wang–Liu's method (15), $a = 0$	Converges to $\alpha = 8$	–	–
Neta–Petkovi'c's method (16), $\lambda = 2$	Converges to $\alpha = 10$	–	–

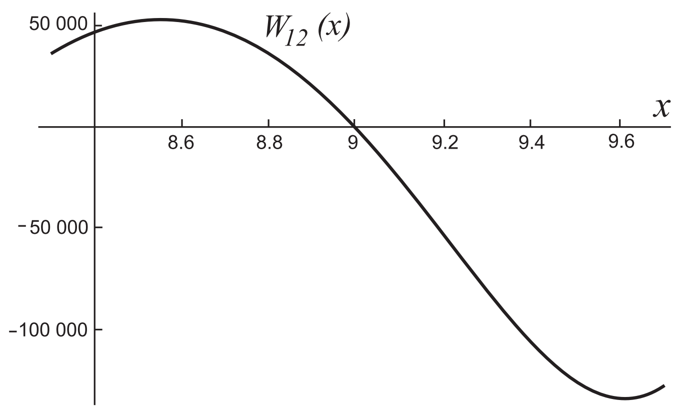

Fig. 1. The graph of Wilkinson-like polynomial W_{12}.

is very close to 8 (to at least the third decimal place) for the tested methods (3) and (12)–(16). This points that practical results coincide well with the theoretical result given in Theorem 1.

After an extensive experimentation, we could not find a specific iterative three-point method in the class of methods (3), (12)–(16) and non-displayed three-point methods of optimal order eight presented in [8–14] which would be asymptotically best for all tested nonlinear functions. A number of numerical examples showed only that one of the tested methods is better for some test-functions, while some other is better for other functions. We could conclude that the convergence behavior of the considered multipoint methods strongly depends on the structure of tested functions and the accuracy of initial approximations.

Regarding the examples given in Table 1 we observe that all tested methods show a very fast convergence in Example 1. The specific methods (3)–(p_i, q_2) are the best in Example 1 but they are not so good in Example 2. Bi–Wu–Ren's method (14) (both variants) produces results of (relatively) small accuracy in Examples 3 and 4. Kung–Traub's method (13) gives worse results in Example 4 than in the remaining examples. In Example 4 the combinations (3)–(p_i, q_3) give the best results, (3)–(p_i, q_1) converge slowly at the beginning of iterative process, while (3)–(p_i, q_2) and (3)–(p_i, q_4) are of the same quality as the methods (13) and (14). The method (15) converges to the root $\alpha = 8$ and the method (16) converges to the root $\alpha = 10$. Note that the polynomial $W_{12}(x)$, considered in Example 4, is of Wilkinson's type. It is known that polynomials of this type are ill-conditioned and take very large values (in magnitude) in the neighborhood of the roots, see Fig. 1, causing that most of root-solvers work with effort in solving equations of this type.

We end this paper with the remark concerning the claims of some authors that their methods are equal or better than existing methods belonging to the same or similar classes. Such assertions are most frequently unjustified from theoretical as well as practical point of view. Actually, it is possible that there are particular examples where some methods work better than the others and vice versa. A more realistic estimate, confirmed by a number of numerical examples, is that multipoint methods without memory of the same order and the same computational cost show a similar convergence behavior and produce results of approximately same accuracy. This equalized behavior is particularly valid when compared methods use Newton-like or Steffensen-like methods in the first step.

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