On additive properties of the Drazin inverse of block matrices and representations*

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Abstract

In this paper, we give a new additive formula for the Drazin inverse under conditions weaker than those used in some current literature on this subject. Also, we obtain representations for the Drazin inverse of a complex block matrix having generalized Schur complement equal to zero.

2000 *Mathematics Subject Classification*: 15A09
Key words: Drazin inverse; block matrix; additive formula

1 Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices and let $A \in \mathbb{C}^{n\times n}$. By $\mathcal{R}(A), \mathcal{N}(A)$ and $\mathrm{rank}(A)$ we denote the range, the null space and the rank of matrix A, respectively. The smallest nonnegative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$, denoted by $\mathrm{ind}(A)$, is called the index of matrix A. If $\mathrm{ind}(A) = k$, then there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies the following relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

The matrix A^d is called the Drazin inverse of A (see [1]). If $\mathrm{ind}(A) = 1$, then the Drazin inverse of A is called the group inverse of A and it is denoted by $A^{\#}$. Clearly, $\mathrm{ind}(A) = 0$ if and only if A is nonsingular, and in that case $A^d = A^{-1}$. In this paper we use notation $A^\pi = I - AA^d$ to denote the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$.

The Drazin inverse of square complex matrices has applications in several areas, such as differential and difference equations, Markov chains and iterative methods (see [2, 3, 4, 5, 6, 7]). For applications of the Drazin inverse of a 2×2 block matrix we refer readers to [2, 8, 9].

*Supported by Grant No. 174007 of the Ministry of Education, Science and Technological Development, Republic of Serbia

Suppose $P, Q \in \mathbb{C}^{n\times n}$. In 1958, Drazin (see [10]) studied the problem of finding the formula for $(P+Q)^d$ and he offered the formula $(P+Q)^d = P^d + Q^d$, which is valid when $PQ = QP = 0$. In the present, there is no formula for $(P+Q)^d$ without any side condition for matrices P and Q, so this problem remains open. However, many authors have considered this problem and provided a formula for $(P+Q)^d$ with some specific conditions for matrices P and Q. Some of them are as follows:

(i) $PQ = 0$ [6];

(ii) $Q^2 = 0$ and $P^2Q = 0$ [11];

(iii) $PQ^2 = 0$ and $P^2Q = 0$ [12];

(iv) $PQ^2 = 0$ and $PQP = 0$ [13].

In this paper we derive a formula for $(P+Q)^d$ under conditions $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ which are weaker than those from previous list.

Another aim–objective of this paper is to derive a representation of the Drazin inverse of 2×2 complex block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where A and D are square matrices, not necessarily of the same size. This problem was firstly posed in 1979 by Campbell and Meyer [3]. According to current literature, there has been no formula for M^d without any side conditions for blocks of matrix M. Special cases of this open problem have been considered, so at present time there are many formulas for M^d under specific conditions for blocks of M. In some papers the expression of M^d is given under conditions which concern the generalized Schur complement of matrix M defined by $S = D - CA^dB$. Here we list some of them:

(i) $CA^\pi = 0$, $A^\pi B = 0$ and $S = 0$ [14];

(ii) $CA^\pi B = 0$, $AA^\pi B = 0$ and $S = 0$ [9];

(iii) $CA^\pi B = 0$, $CA^\pi A = 0$ and $S = 0$ [9];

(iv) $CA^\pi BC = 0$, $AA^\pi BC = 0$ and $S = 0$ [13];

(v) $BCA^\pi B = 0$, $BCA^\pi A = 0$ and $S = 0$ [13];

(vi) $ABCA^\pi = 0$, BCA^π is nilpotent and $S = 0$ [11];

(vii) $A^\pi BCA = 0$, $A^\pi BC$ is nilpotent and $S = 0$ [11];

(viii) $ABCA^\pi = 0$, $A^\pi ABC = 0$ and $S = 0$ [15];

(ix) $ABCA^\pi = 0$, $CBCA^\pi = 0$ and $S = 0$ [15].

In this paper we derive some new representations for M^d, which generalizes representations given under conditions from previous list.

Before we give our main results, we state some auxiliary lemmas as follows.

Lemma 1.1 [1] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. *Then* $(AB)^d = A((BA)^2)^d B$.

Lemma 1.2 [6] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If* $PQ = 0$ *then*

$$(P+Q)^d = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1} + \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi.$$

Lemma 1.3 [14] *Let* M *be a matrix of the form* (1.1)*, such that* $S = 0$. *If* $A^\pi B = 0$ *and* $CA^\pi = 0$*, then*

$$M^d = \begin{bmatrix} I \\ CA^d \end{bmatrix} \left((AW)^d\right)^2 A \begin{bmatrix} I & A^d B \end{bmatrix},$$

where $W = AA^d + A^d BCA^d$.

2 Additive results

In this section we investigate the Drazin inverse of the sum of two matrices. The following theorem is the main tool in our sequel development.

Theorem 2.1 *Let* $P, Q \in \mathbb{C}^{n\times n}$. *If* $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ *and* $PQ^3 = 0$ *then*

$$\begin{aligned} (P+Q)^d \;=\; & \Bigg(\sum_{i=0}^{\mathrm{ind}((P+Q)Q)-1} ((P+Q)Q)^\pi ((P+Q)Q)^i (((P+Q)P)^d)^{i+1} \\ & + \sum_{i=0}^{\mathrm{ind}((P+Q)P)-1} (((P+Q)Q)^d)^{i+1} ((P+Q)P)^i ((P+Q)P)^\pi \Bigg) (P+Q), \end{aligned}$$

where for $n \in \mathbb{N}$

$$\begin{aligned} (((P+Q)P)^d)^n &= \sum_{i=0}^{\mathrm{ind}(QP)-1} (QP)^\pi (QP)^i (P^d)^{2(i+n)} + \sum_{i=0}^{\mathrm{ind}(P^2)-1} ((QP)^d)^{i+n} P^{2i} P^\pi \\ &\quad - \sum_{i=1}^{n-1} ((QP)^d)^i (P^d)^{2(n-i)}, \\ (((P+Q)Q)^d)^n &= \sum_{i=0}^{\mathrm{ind}(Q^2)-1} Q^\pi Q^{2i} ((PQ)^d)^{i+n} + \sum_{i=0}^{\mathrm{ind}(PQ)-1} (Q^d)^{2(i+n)} (PQ)^i (PQ)^\pi \\ &\quad - \sum_{i=1}^{n-1} (Q^d)^{2i} ((PQ)^d)^{n-i}, \end{aligned}$$

and

$$
\begin{aligned}
((P+Q)P)^{\pi} &= (QP)^{\pi}P^{\pi} - \sum_{i=0}^{\mathrm{ind}(QP)-2} (QP)^{\pi}(QP)^{i+1}(P^d)^{2(i+1)} \\
&- \sum_{i=0}^{\mathrm{ind}(P^2)-2} ((QP)^d)^{i+1}P^{2(i+1)}P^{\pi}, \\
((P+Q)Q)^{\pi} &= Q^{\pi}(PQ)^{\pi} - \sum_{i=0}^{\mathrm{ind}(Q^2)-2} Q^{\pi}Q^{2(i+1)}((PQ)^d)^{i+1} \\
&- \sum_{i=0}^{\mathrm{ind}(PQ)-2} (Q^d)^{2(i+1)}(PQ)^{i+1}(PQ)^{\pi}.
\end{aligned}
$$

Proof. Using Lemma 1.1, we have that $(P+Q)^d = (P+Q)((P+Q)^d)^2 = (P+Q)(P(P+Q)+Q(P+Q))^d$. Denote by $F = P(P+Q)$ and $G = Q(P+Q)$. Since $FG = 0$, matrices F and G satisfy the condition of Lemma 1.2 and therefore

$$
(P+Q)^d = (P+Q)\left(\sum_{i=0}^{\mathrm{ind}(G)-1} G^{\pi}G^i(F^d)^{i+1} + \sum_{i=0}^{\mathrm{ind}(F)-1} (G^d)^{i+1}F^iF^{\pi}\right).
$$

Furthermore, by Lemma 1.1 we have $F^d = P(((P+Q)P)^d)^2(P+Q)$ and $G^d = Q(((P+Q)Q)^d)^2(P+Q)$. If we denote by $F_1 = (P+Q)P$ and $G_1 = (P+Q)Q$, we get $F^d = P(F_1^d)^2(P+Q)$ and $G^d = Q(G_1^d)^2(P+Q)$. Moreover,

$$
(F^d)^n = P(F_1^d)^{n+1}(P+Q),\ (G^d)^n = Q(G_1^d)^{n+1}(P+Q),
$$

for every $n \in \mathbb{N}$. After some computations we get

$$
\begin{aligned}
(P+Q)^d = \quad & (P+Q)\Bigg(P(F_1^d)^2 - QG_1^dF_1^d + \sum_{i=0}^{\mathrm{ind}(G_1)-1} QG_1^{\pi}G_1^i(F_1^d)^{i+2} \\
& + \sum_{i=0}^{\mathrm{ind}(F_1)-1} Q(G_1^d)^{i+2}F_1^iF_1^{\pi}\Bigg)(P+Q) \\
= \quad & \left(\sum_{i=0}^{\mathrm{ind}(G)-1} G^{\pi}G^i(F^d)^{i+1} + \sum_{i=0}^{\mathrm{ind}(F)-1} (G^d)^{i+1}F^iF^{\pi}\right)(P+Q). \qquad (2.1)
\end{aligned}
$$

Notice that $F_1 = P^2 + QP$. Since $P^2QP = 0$, matrices P^2 and QP satisfy condition of Lemma 1.2. After applying Lemma 1.2 we obtain

$$
\begin{aligned}
(F_1^d)^n = & \sum_{i=0}^{\mathrm{ind}(QP)-1} (QP)^{\pi}(QP)^i(P^d)^{2(i+n)} + \sum_{i=0}^{\mathrm{ind}(P^2)-1} ((QP)^d)^{i+n}P^{2i}P^{\pi} \\
& - \sum_{i=1}^{n-1}((QP)^d)^i(P^d)^{2(n-i)},
\end{aligned} \qquad (2.2)
$$

for every $n \in \mathbb{N}$. Similarly, from $PQ^3 = 0$ and Lemma 1.2, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}(G_1^d)^n = & \sum_{i=0}^{\mathrm{ind}(Q^2)-1} Q^\pi Q^{2i}((PQ)^d)^{i+n} + \sum_{i=0}^{\mathrm{ind}(PQ)-1} (Q^d)^{2(i+n)}(PQ)^i(PQ)^\pi \\ & - \sum_{i=1}^{n-1}(Q^d)^{2i}((PQ)^d)^{n-i}.\end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) we get that the statement of the theorem is valid.□

The next theorem is a symmetrical formulation of Theorem 2.1.

Theorem 2.2 *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQP^2 = 0$, $Q^2P^2 = 0$, $PQ^2P = 0$ and $Q^3P = 0$ then*

$$\begin{aligned}(P+Q)^d &= (P+Q)\left(\sum_{i=0}^{\mathrm{ind}(Q(P+Q))-1} ((P(P+Q))^d)^{i+1}(Q(P+Q))^i(Q(P+Q))^\pi\right. \\ &+ \left.\sum_{i=0}^{\mathrm{ind}(P(P+Q))-1} (P(P+Q))^\pi(P(P+Q))^i((Q(P+Q))^d)^{i+1}\right),\end{aligned}$$

where for $n \in \mathbb{N}$

$$\begin{aligned}((P(P+Q))^d)^n &= \sum_{i=0}^{\mathrm{ind}(PQ)-1} (P^d)^{2(i+n)}(PQ)^i(PQ)^\pi + \sum_{i=0}^{\mathrm{ind}(P^2)-1} P^\pi P^{2i}((PQ)^d)^{i+n} \\ &- \sum_{i=1}^{n-1}(P^d)^{2(n-i)}((PQ)^d)^i, \\ ((Q(P+Q))^d)^n &= \sum_{i=0}^{\mathrm{ind}(Q^2)-1} ((QP)^d)^{i+n}Q^{2i}Q^\pi + \sum_{i=0}^{\mathrm{ind}(QP)-1} (QP)^\pi(QP)^i(Q^d)^{2(i+n)} \\ &- \sum_{i=1}^{n-1}((QP)^d)^{n-i}(Q^d)^{2i},\end{aligned}$$

and

$$\begin{aligned}(P(P+Q))^\pi &= P^\pi(PQ)^\pi - \sum_{i=0}^{\mathrm{ind}(PQ)-2} (P^d)^{2(i+1)}(PQ)^{i+1}(PQ)^\pi \\ &- \sum_{i=0}^{\mathrm{ind}(P^2)-2} P^\pi P^{2(i+1)}((PQ)^d)^{i+1},\end{aligned}$$

$$
\begin{aligned}
(Q(P+Q))^{\pi} &= (QP)^{\pi}Q^{\pi} - \sum_{i=0}^{\mathrm{ind}(Q^2)-2} ((QP)^d)^{i+1}Q^{2(i+1)}Q^{\pi} \\
&- \sum_{i=0}^{\mathrm{ind}(QP)-2} (QP)^{\pi}(QP)^{i+1}(Q^d)^{2(i+1)}.
\end{aligned}
$$

Notice that one special case of Theorem 2.1 is when matrices P and Q satisfy the conditions $P^2QP = 0$ and $PQ^2 = 0$. Similarly, a special case of Theorem 2.2 is when $PQP^2 = 0$ and $Q^2P = 0$ is valid. The following additive formulas are corollaries of these cases, respectively, which we will use in section 3 to obtain representations for the Drazin inverse of block matrix.

Corollary 2.1 *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^2QP = 0$ and $Q^2 = 0$, then*

$$
\begin{aligned}
(P+Q)^d = \Bigg(&\sum_{i=0}^{r-1} \left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right) P^{2i}P^{\pi} \\
&+ \sum_{i=0}^{s-1} \left((PQ)^{\pi}(PQ)^i + (QP)^{\pi}(QP)^i\right)(P^d)^{2(i+1)} - (P^d)^2\Bigg)(P+Q),
\end{aligned}
$$

where $r = \mathrm{ind}(P^2)$ and $s = max\,\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.

Corollary 2.2 *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQP^2 = 0$ and $Q^2 = 0$, then*

$$
\begin{aligned}
(P+Q)^d =&(P+Q)\Bigg(\sum_{i=0}^{r-1} P^{\pi}P^{2i}\left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1}\right) \\
&+ \sum_{i=0}^{s-1}(P^d)^{2(i+1)}\left((PQ)^i(PQ)^{\pi} + (QP)^i(QP)^{\pi}\right) - (P^d)^2\Bigg),
\end{aligned}
$$

where $r = \mathrm{ind}(P^2)$ and $s = max\,\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.

3 Representations for the Drazin inverse of block matrix

Through this section we assume that matrix M is defined by (1.1), where A and D are square matrices and generalized Schur complement $S = D - CA^dB$ of matrix M is equal to zero.

In [14] Miao offered a representation for M^d under conditions $CA^{\pi} = 0$ and $A^{\pi}B = 0$. This result was generalized in [9], where authors gave the formula for M^d under conditions $CA^{\pi}A = 0$ and $CA^{\pi}B = 0$. Yang and Liu [13] extended this result and derived the representation for M^d when $BCA^{\pi}A = 0$ and $BCA^{\pi}B = 0$ holds. The following theorem is a generalization of this result.

Theorem 3.1 *Let M be a matrix of the form (1.1) such that $S = 0$. If $ABCA^\pi A = 0$ and $ABCA^\pi B = 0$, then*

$$M^d = \left(\begin{bmatrix} (BCA^\pi)^\pi & 0 \\ -(CA^\pi B)^d CA^\pi A & (CA^\pi B)^\pi \end{bmatrix} (P^d)^2 \right.$$
$$+ \sum_{i=0}^{t-1} \begin{bmatrix} (BCA^\pi)^\pi (BCA^\pi)^{i+1} & 0 \\ (CA^\pi B)^\pi (CA^\pi B)^i CA^\pi A & (CA^\pi B)^\pi (CA^\pi B)^{i+1} \end{bmatrix} (P^d)^{2i+4}$$
$$\left. + \sum_{i=0}^{r-1} \begin{bmatrix} ((BCA^\pi)^d)^{i+1} & 0 \\ ((CA^\pi B)^d)^{i+2} CA^\pi A & ((CA^\pi B)^d)^{i+1} \end{bmatrix} P^{2i} P^\pi \right) M,$$

where

$$P = \begin{bmatrix} A & B \\ CA^d A & CA^d B \end{bmatrix},$$
$$(P^d)^n = \left(I + \sum_{j=0}^{l-1} \begin{bmatrix} 0 & A^j A^\pi B \\ 0 & 0 \end{bmatrix} (P_1^d)^{j+1} \right) (P_1^d)^n,$$
$$(P_1^d)^n = \begin{bmatrix} I \\ CA^d \end{bmatrix} ((AW)^d)^{n+1} A \begin{bmatrix} I & A^d B \end{bmatrix}, \ W = AA^d + A^d BCA^d,$$

for every $n \in \mathbb{N}$, and $r = \mathrm{ind}(P^2)$, $l = \mathrm{ind}(A)$, $t = max\,\{\mathrm{ind}(CA^\pi B), \mathrm{ind}(BCA^\pi) - 1\}$.

Proof. Consider the splitting of matrix M

$$M = \begin{bmatrix} A & B \\ C & CA^d B \end{bmatrix} = \begin{bmatrix} A & B \\ CA^d A & CA^d B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix}.$$

If we denote by $P = \begin{bmatrix} A & B \\ CA^d A & CA^d B \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix}$, we have that $P^2 QP = 0$ and $Q^2 = 0$. Hence, the conditions of Corollary 2.1 are satisfied and

$$(P+Q)^d = \left(\sum_{i=0}^{r-1} \left(((PQ)^d)^{i+1} + ((QP)^d)^{i+1} \right) P^{2i} P^\pi \right.$$
$$\left. + \sum_{i=0}^{s-1} \left((PQ)^\pi (PQ)^i + (QP)^\pi (QP)^i \right) (P^d)^{2(i+1)} - (P^d)^2 \right) M, \quad (3.1)$$

where $r = \mathrm{ind}(P^2)$ and $s = max\,\{\mathrm{ind}(PQ), \mathrm{ind}(QP)\}$.
Obviously $Q^d = 0$ and $Q^\pi = I$. If we split matrix P as

$$P = \begin{bmatrix} A & B \\ CA^d A & CA^d B \end{bmatrix} = \begin{bmatrix} A^2 A^d & AA^d B \\ CA^d A & CA^d B \end{bmatrix} + \begin{bmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{bmatrix},$$

and denote by $P_1 = \begin{bmatrix} A^2 A^d & AA^d B \\ CA^d A & CA^d B \end{bmatrix}$, $P_2 = \begin{bmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{bmatrix}$, we get $P_1 P_2 = 0$ and P_2 is $(l+1)$-nilpotent. After using Lemma 1.2 we get

$$(P^d)^n = \left(I + \sum_{i=0}^{l-1} P_2^{i+1} (P_1^d)^{i+1} \right) (P_1^d)^n,$$

for $n \in \mathbb{N}$. Notice that matrix P_1 satisfy conditions of Lemma 1.3, so after applying it we obtain

$$(P_1^d)^n = \begin{bmatrix} I \\ CA^d \end{bmatrix} ((AW)^d)^{n+1} A \begin{bmatrix} I & A^d B \end{bmatrix}.$$

Therefore,

$$(P^d)^n = \left(I + \sum_{j=0}^{l-1} \begin{bmatrix} 0 & A^i A^\pi B \\ 0 & 0 \end{bmatrix} (P_1^d)^{j+1} \right) (P_1^d)^n. \tag{3.2}$$

After computation we get:

$$(PQ)^n = \begin{cases} \begin{bmatrix} BCA^\pi & 0 \\ CA^d BCA^\pi & 0 \end{bmatrix}, & \text{if } n = 1 \\ \begin{bmatrix} (BCA^\pi)^i & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } n \geq 2 \end{cases},$$

$$((PQ)^d)^n = \begin{bmatrix} ((BCA^\pi)^d)^n & 0 \\ 0 & 0 \end{bmatrix}, \ (PQ)^\pi = \begin{bmatrix} (BCA^\pi)^\pi & 0 \\ 0 & I \end{bmatrix},$$

$$(QP)^n = \begin{bmatrix} 0 & 0 \\ (CA^\pi B)^{n-1} CA^\pi A & (CA^\pi B)^n \end{bmatrix},$$

$$((QP)^d)^n = \begin{bmatrix} 0 & 0 \\ ((CA^\pi B)^d)^{n+1} CA^\pi A & ((CA^\pi B)^d)^n \end{bmatrix},$$

$$(QP)^\pi = \begin{bmatrix} I & 0 \\ -(CA^\pi B)^d CA^\pi A & (CA^\pi B)^\pi \end{bmatrix}.$$

After substituting this expressions and (3.2) into (3.1) we complete the proof. □

Remark 1 *Bu et al. offered formulas for M^d under conditions $ABCA^\pi = 0$, $A^\pi ABC = 0$ [15, Theorem 4.1] and under conditions $ABCA^\pi = 0$, $CBCA^\pi = 0$ [15, Theorem 4.3]. In [11, Theorem 3.3] the representation for M^d is given under conditions $ABCA^\pi = 0$, BCA^π is nilpotent. We remark that a special case of Theorem 3.1 is when blocks of matrix M satisfy the condition $ABCA^\pi = 0$. Therefore the conditions $A^\pi ABC = 0$ from [15, Theorem 4.1], $CBCA^\pi = 0$ from [15, Theorem 4.3] and BCA^π is nilpotent from [11, Theorem 3.3] are superfluous.*

The next theorem is an extension of a case when $CA^\pi BC = 0$ and $AA^\pi BC = 0$ hold, which was studied by Yang and Liu [13].

Theorem 3.2 *Let M be a matrix defined by (1.1), such that $S = 0$. If $AA^\pi BCA = 0$ and $CA^\pi BCA = 0$, then*

$$\begin{aligned} M^d = & M \Bigg((P^d)^2 \begin{bmatrix} (A^\pi BC)^\pi & -AA^\pi B (CA^\pi B)^d \\ 0 & (CA^\pi B)^\pi \end{bmatrix} \\ & + \sum_{i=0}^{t-1} (P^d)^{2i+4} \begin{bmatrix} (A^\pi BC)^{i+1} (A^\pi BC)^\pi & AA^\pi B (CA^\pi B)^i (CA^\pi B)^\pi \\ 0 & (CA^\pi B)^{i+1} (CA^\pi B)^\pi \end{bmatrix} \\ & + \sum_{i=0}^{r-1} P^\pi P^{2i} \begin{bmatrix} ((A^\pi BC)^d)^{i+1} & AA^\pi B ((CA^\pi B)^d)^{i+2} \\ 0 & ((CA^\pi B)^d)^{i+1} \end{bmatrix} \Bigg), \end{aligned}$$

where

$$P = \begin{bmatrix} A & AA^dB \\ C & CA^dB \end{bmatrix},$$

$$(P^d)^n = (P_1^d)^n \left(I + \sum_{j=0}^{l-1} (P_1^d)^{j+1} \begin{bmatrix} 0 & 0 \\ CA^jA^\pi & 0 \end{bmatrix} \right),$$

$$(P_1^d)^n = \begin{bmatrix} I \\ CA^d \end{bmatrix} ((AW)^d)^{n+1} A \begin{bmatrix} I & A^dB \end{bmatrix}, \; W = AA^d + A^dBCA^d,$$

for every $n \in \mathbb{N}$*, and* $r = \text{ind}(P^2)$*,* $l = \text{ind}(A)$*,* $t = max\,\{\text{ind}(CA^\pi B), \text{ind}(A^\pi BC) - 1\}$*.*

Proof. We can split matrix M as

$$M = \begin{bmatrix} A & B \\ C & CA^dB \end{bmatrix} = \begin{bmatrix} A & AA^dB \\ C & CA^dB \end{bmatrix} + \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}.$$

If we denote by $P = \begin{bmatrix} A & AA^dB \\ C & CA^dB \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}$, we have that matrices P and Q satisfy Corollary 2.2. Using similar method as in Theorem 3.1 we get that the statement of the theorem is true. □

Remark 2 *Martínez–Serrano and Castro-González derived a formula for* M^d *under conditions* $A^\pi BCA = 0$ *and* $A^\pi BC$ *is nilpotent [11, Corollary 3.4]. Notice that Theorem 3.2 is an extension of a case when* $A^\pi BCA = 0$*. Hence, the condition* $A^\pi BC$ *is nilpotent from [11, Corollary 3.4] is superfluous.*

4 Numerical example

In this section we give a numerical example to demonstrate the application of Theorem 3.1.

Example Consider the block matrix M of a form (1.1) , where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By computing we get that generalized Schur complement $S = D - CA^dB$ is equal to zero and $ABCA^\pi = 0$. Since $A^\pi ABC \neq 0$, $CBCA^\pi \neq 0$ and matrix BCA^π is not nilpotent, formulas for M^d from [15, Theorem 4.1], [15, Theorem 4.3] and [11, Theorem 3.3] fail to apply. However, the conditions of Theorem 3.1 are satisfied, so we can apply it.

We have that $\text{ind}(A) = 2$ and

$$A^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also, we get that $\text{ind}(P) = 3$ and

$$P^d = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After applying Theorem 3.1, we get

$$M^d = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Acknowledgements

The author would like to thank the anonymous referees for their relevant and useful comments, which helped to improve the paper.

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