Additive results for the Drazin inverse of block matrices and applications

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Abstract

In this paper we consider the Drazin inverse of a sum of two matrices and we derive additive formulas under conditions weaker than those used in some recent papers on the subject. Like a corollary we get the main results from the paper of H. Yang, X. Liu (*The Drazin inverse of the sum of two matrices and its applications*, Journ.Comp.Appl.Math., 235 (2011) 1412–1417). As an application we give some new representations for the Drazin inverse of a block matrix.

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1 Introduction

Let A be a square complex matrix. We denote by $\mathcal{R}(A), \mathcal{N}(A)$ and $\mathrm{rank}(A)$, the range, the null space and the rank of matrix A, respectively. In addition, by $\mathrm{ind}(A)$ we denote the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$, called the index of A. For every matrix $A \in \mathbb{C}^{n\times n}$, such that $\mathrm{ind}(A) = k$, there exists an unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies the relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

The matrix A^d is called the Drazin inverse of A (see [9, 10]). The case $\mathrm{ind}(A) = 0$ is valid if and only if A is nonsingular, so A^d reduces to A^{-1}.

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By $A^\pi = I - AA^d$ we denote the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$. If the lower limit of a sum is greater than its upper limit, we always define the sum to be 0. For example, the sum $\sum_{l=1}^{k-2} * = 0$, for $k \leq 2$. We agree that $A^0 = I$, for any matrix A.

Let $P, Q \in \mathbb{C}^{n\times n}$. The open problem of finding explicit formulas for the Drazin inverse of $P + Q$ in terms of P, Q, P^d, Q^d was posed by Drazin in 1958 [9]. Many authors have considered this problem and have provided formulas for $(P + Q)^d$ under some specific conditions for the matrices P and Q . Some of them are listed bellow:

(i) $PQ = QP = 0$ [9];

(ii) $PQ = 0$ [11];

(iii) $P^2Q = 0$ and $PQ^2 = 0$ [2];

(iv) $PQ^2 = 0$ and $PQP = 0$ [13].

In Section 2 we derive some formulas for $(P + Q)^d$ under weaker conditions than those given in [9, 11, 2, 13].

Formulas for $(P + Q)^d$ can be very useful for deriving formulas for the Drazin inverse of a 2×2 block matrix. Actually, in 1979 Campbell and Meyer[4] posed the problem of finding an explicit representation for the Drazin inverse of a complex block matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

in terms of its blocks, where A and D are square matrices, not necessarily of the same size. No formula for M^d has yet been offered without any restrictions upon the blocks. Special cases of this problem have been studied, so at present time we have some formulas for M^d under certain conditions on the blocks of M. In Section 3 we derive some new formulas for M^d. These results are generalizations of some of the results from [7, 8].

First, we will state some auxiliary lemmas.

Lemma 1.1 [1] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. Then $(AB)^d = A((BA)^2)^d B$.*

Lemma 1.2 [12] *Let M_1 and M_2 be matrices of a form*

$$M_1 = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}, \; M_2 = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix}$$

where A *and* B *are square matrices such that* $\operatorname{ind}(A) = r$*,* $\operatorname{ind}(B) = s$*. Then* $\max\{r, s\} \leq \operatorname{ind}(M_i) \leq r + s$*,* $i = 1, 2$*, and*

$$M_1^d = \begin{bmatrix} A^d & 0 \\ S & B^d \end{bmatrix}, \; M_2^d = \begin{bmatrix} B^d & S \\ 0 & A^d \end{bmatrix},$$

where

$$S = (B^d)^2 \left(\sum_{i=0}^{r-1} (B^d)^i C A^i \right) A^\pi + B^\pi \left(\sum_{i=0}^{s-1} B^i C (A^d)^i \right) (A^d)^2 - B^d C A^d.$$

2 The Drazin inverse of a sum of two matrices

Let us define for $j \in \mathbb{N}$, the set $U_j = \{(p_1, q_1, p_2, q_2, ..., p_j, q_j) : \sum_{i=1}^{j} p_i + \sum_{i=1}^{j} q_i = j - 1, \; p_i, q_i \in \{0, 1, ..., j - 1\}, \; i = \overline{1, j}\}$.

Theorem 2.1 *Let* $P, Q \in \mathbb{C}^{n \times n}$ *be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$ *and* $k \in \mathbb{N}$*. If*

$$PQ \prod_{i=1}^{k} (P^{p_i} Q^{q_i}) = 0, \tag{2.1}$$

for every $(p_1, q_1, p_2, q_2, ..., p_k, q_k) \in U_k$*, then*

$$\begin{aligned} (P+Q)^d &= Y_1 + Y_2 \\ &+ \sum_{i=1}^{k-1} \Big(Y_1 (P^d)^{i+1} + (Q^d)^{i+1} Y_2 - \sum_{j=1}^{i+1} (Q^d)^j (P^d)^{i+2-j} \Big) PQ (P+Q)^{i-1}, \end{aligned} \tag{2.2}$$

where

$$Y_1 = Q^\pi \left(\sum_{i=0}^{s-1} Q^i (P^d)^i \right) P^d, \; Y_2 = Q^d \left(\sum_{i=0}^{r-1} (Q^d)^i P^i \right) P^\pi. \tag{2.3}$$

Proof : We will prove this result using mathematical induction on k. For $k = 1$ the theorem is true (see [11]). Now, we will assume that it holds for $k - 1$ and let us prove that it holds for k.

Using Lemma 1.1 we have that

$$
\begin{aligned}
(P+Q)^d &= \begin{bmatrix} I & Q \end{bmatrix} \left(\left(\begin{bmatrix} P \\ I \end{bmatrix} \begin{bmatrix} I & Q \end{bmatrix} \right)^2 \right)^d \begin{bmatrix} P \\ I \end{bmatrix} \\
&= \begin{bmatrix} I & Q \end{bmatrix} \begin{bmatrix} P^2+PQ & P^2Q+PQ^2 \\ P+Q & PQ+Q^2 \end{bmatrix}^d \begin{bmatrix} P \\ I \end{bmatrix}.
\end{aligned}
$$

Denote by

$$
M = \begin{bmatrix} P^2+PQ & P^2Q+PQ^2 \\ P+Q & PQ+Q^2 \end{bmatrix}, M_1 = \begin{bmatrix} PQ & P^2Q+PQ^2 \\ 0 & PQ \end{bmatrix} \text{ and }
$$

$$
M_2 = \begin{bmatrix} P^2 & 0 \\ P+Q & Q^2 \end{bmatrix}.
$$

By computation, we show that for arbitrary $n \in \mathbb{N}$,

$$
M_1^n = \begin{bmatrix} (PQ)^n & W_n \\ 0 & (PQ)^n \end{bmatrix} \text{ and } M_2^n = \begin{bmatrix} P^{2n} & 0 \\ S_n & Q^{2n} \end{bmatrix}, \tag{2.4}
$$

where $W_n = \sum_{i=0}^{n-1}(PQ)^i P(P+Q)Q(PQ)^{n-1-i}$ and $S_n = \sum_{i=0}^{n-1} Q^{2i}(P+Q)P^{2(n-1-i)}$.

It is evident that $M_1^n = 0$, for every $n \geq \dfrac{k+1}{2}$. Also, by straightforward computation we have that

$$
M_1 M_2 \prod_{i=1}^{k-1}(M_1^{p_i} M_2^{q_i}) = 0,
$$

for every $(p_1, q_1, p_2, q_2, ..., p_{k-1}, q_{k-1}) \in U_{k-1}$. Hence, M_1 and M_2 satisfy the conditions of the theorem for $k-1$. By induction hypothesis we get that

$$
(M_1+M_2)^d = Z_1 + Z_2
$$

$$
+\sum_{i=1}^{k-2}\Big(Z_1(M_1^d)^{i+1} + (M_2^d)^{i+1}Z_2 - \sum_{j=1}^{i+1}(M_2^d)^j(M_1^d)^{i+2-j}\Big)M_1M_2(M_1+M_2)^{i-1},
$$

where Z_1 and Z_2 are defined by (2.3), in function of matrices M_1 and M_2.

Since $M_1^d = 0$ and $M_1^\pi = I$, we get that $Z_1 = 0$ and

$$
Z_2 = \sum_{i=0}^{[\frac{k+1}{2}]-1} (M_2^d)^{i+1} M_1^i.
$$

Therefore we get

$$(M_1 + M_2)^d = Z_2 + \sum_{i=1}^{k-2} \Big((M_2^d)^{i+1} Z_2\Big) M_1 M_2 (M_1 + M_2)^{i-1}. \tag{2.5}$$

We have that

$$(M_2^d)^n = \begin{bmatrix} (P^d)^{2n} & 0 \\ A_n & (Q^d)^{2n} \end{bmatrix}, \tag{2.6}$$

where $A_n = Y_1(P^d)^{2n} + (Q^d)^{2n} Y_2 - \sum_{i=1}^{2n} (Q^d)^i (P^d)^{2n+1-i}$, for any $n \in \mathbb{N}$. Also, we get that

$$(M_1 + M_2)^n = \begin{bmatrix} P(P+Q)^{2n-1} & P(P+Q)^{2n-1}Q \\ (P+Q)^{2n-1} & (P+Q)^{2n-1}Q \end{bmatrix}, \tag{2.7}$$

for all $n \in \mathbb{N}$. Substituting (2.4), (2.6) and (2.7) into (2.5) it completes the proof.□

As corollary of Theorem 2.1 in the case $k = 1$, we get the main result from [11].

Corollary 2.1 [11] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If* $PQ = 0$ *then*

$$(P + Q)^d = Y_1 + Y_2,$$

where Y_1 *and* Y_2 *are defined by* (2.3).

If we consider the case where $k = 2$ in Theorem 2.1 we obtain as a corollary the main result in [13].

Corollary 2.2 [13] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If* $PQP = 0$ *and* $PQ^2 = 0$ *then*

$$(P + Q)^d = Y_1 + Y_2 + \left(Y_1 (P^d)^2 + (Q^d)^2 Y_2 - \sum_{i=1}^{2} (Q^d)^i (P^d)^{3-i} \right) PQ.$$

where Y_1 *and* Y_2 *are defined by* (2.3).

When $k = 3$, we get the following new result.

Corollary 2.3 *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If $PQP^2 = 0$, $PQPQ = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ then*

$$\begin{aligned}(P+Q)^d &= Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2Y_2 - \sum_{i=1}^{2}(Q^d)^i(P^d)^{3-i}\right)PQ \\ &\quad + \left(Y_1(P^d)^3 + (Q^d)^3Y_2 - \sum_{i=1}^{3}(Q^d)^i(P^d)^{4-i}\right)(PQP + PQ^2),\end{aligned}$$

where Y_1 and Y_2 are defined by (2.3).

3 Applications

Let M be a matrix of the form (1.1), where A and D are square matrices not necessarily of the same size. Throughout this section we assume that $\mathrm{ind}(A) = r$ and $\mathrm{ind}(D) = s$.

The problem of finding M^d was studied in [7], where the authors gave a representation for M^d under the assumptions $BC = 0$, $BD = 0$ and $DC = 0$. This case was extended to the case when $BC = 0$ and $DC = 0$ (see [6]), and also to a case $BC = 0$, $BDC = 0$ and $BD^2 = 0$ (see [8]). In the next theorem we derive an explicit representation of M^d, which is an extension of a case when $BC = 0$ and $BD = 0$.

Using the special case of Theorem 2.1 when $k = 3$, we get the following result.

Theorem 3.1 *Let M be given by* (1.1). *If $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$, then*

$$M^d = \begin{bmatrix} A^d + B\Sigma_1 + ((A^d)^3 + B\Sigma_3)BC & \begin{matrix}(A^d)^2B + B(D^d)^2 + B\Sigma_2B \\ +B\Sigma_3BD\end{matrix} \\ \Sigma_0 + \Sigma_2BC & D^d + \Sigma_1B + \Sigma_2BD \end{bmatrix},$$

where

$$\begin{aligned}\Sigma_k = &\ (D^d)^2\sum_{i=0}^{r-1}(D^d)^{i+k}CA^iA^\pi + D^\pi\sum_{i=0}^{s-1}D^iC(A^d)^{i+k}(A^d)^2 \\ &- \sum_{i=0}^{k}(D^d)^{i+1}C(A^d)^{k-i+1},\ k \geq 0.\end{aligned} \tag{3.1}$$

Proof. If we split matrix M as

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} := P + Q,$$

we have that $Q^2 = 0$, $PQP^2 = 0$ and $PQPQ = 0$. Hence, matrices P and Q satisfy the conditions of Corollary 2.3 and we get

$$M^d = P^d + Q(P^d)^2 + (P^d)^2 Q + Q(P^d)^3 Q + (P^d)^3 QP + Q(P^d)^4 QP. \quad (3.2)$$

Now, by Lemma 1.2, we have that for any $k \geq 1$,

$$(P^d)^k = \begin{bmatrix} (A^d)^k & 0 \\ \Sigma_{k-1} & (D^d)^k \end{bmatrix}.$$

After computing all elements of the sum (3.2), we get that the statement of this theorem is valid. □

Corollary 3.1 [8] *If M is matrix of a form* (1.1)*, such that $BC = 0$ and $BD = 0$, then*

$$M^d = \begin{bmatrix} A^d & (A^d)^2 B \\ \Sigma_0 & D^d + \Sigma_1 B \end{bmatrix},$$

where Σ_k, $(k \geq 0)$ is defined by (3.1).

In the next theorem we give an extension of a representation for M^d, which is proved in [5].

Theorem 3.2 *If matrix M, defined by* (1.1)*, is such that $BCA = 0$, $DCA = 0$, $CBC = 0$ and $CBD = 0$, then*

$$M^d = \begin{bmatrix} A^d + Z_1 C + Z_2 CA & Z_0 + Z_2 CB \\ (D^d)^2 C + C(A^d)^2 + CZ_2 C + CZ_3 CA & D^d + CZ_1 + ((D^d)^3 + CZ_3)CB \end{bmatrix},$$

where

$$\begin{aligned} Z_k = \; & (A^d)^2 \sum_{i=0}^{s-1} (A^d)^{i+k} B D^i D^\pi + A^\pi \sum_{i=0}^{r-1} A^i B (D^d)^{i+k} (D^d)^2 \\ & - \sum_{i=0}^{k} (A^d)^{i+1} B (D^d)^{k-i+1}, \; k \geq 0. \end{aligned} \quad (3.3)$$

Proof. Using the splitting of matrix M

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} := P + Q$$

we get that matrices P and Q satisfy the conditions of Corollary 2.3. Therefore

$$M^d = P^d + Q(P^d)^2 + (P^d)^2Q + Q(P^d)^3Q + (P^d)^3QP + Q(P^d)^4QP. \quad (3.4)$$

Using Lemma 1.2, we get

$$(P^d)^k = \begin{bmatrix} (A^d)^k & Y_{k-1} \\ 0 & (D^d)^k \end{bmatrix}, \quad (3.5)$$

where $k \geq 1$. Substituting (3.5) into (3.4) completes the proof. □

As a corollary of Theorem 3.2, we have the following result.

Corollary 3.2 [5] *Let M be given by* (1.1) *and let $CA = 0$ and $CB = 0$. Then*

$$M^d = \begin{bmatrix} A^d + Z_1C & Z_0 \\ (D^d)^2C & D^d \end{bmatrix},$$

where Z_k, $(k \geq 0)$ is defined by (3.3).

In [8] authors gave an explicit representation for M^d under conditions $BD^\pi C = 0$, $BDD^d = 0$ and $DD^\pi C = 0$. Here we replace the last condition by the two weaker conditions $DD^\pi CA = 0$ and $DD^\pi CB = 0$.

Theorem 3.3 *Let M be given by* (1.1). *If $BD^\pi C = 0$, $BDD^d = 0$, $DD^\pi CA = 0$ and $DD^\pi CB = 0$, then*

$$M^d = \begin{bmatrix} A^d + \sum_{i=0}^{s-1}(A^d)^{i+3}BD^iC & \sum_{i=0}^{s-1}(A^d)^{i+2}BD^i \\ \Phi_0 + \sum_{i=0}^{s-1}\Phi_{i+2}BD^iC & D^d + \sum_{i=0}^{s-1}\Phi_{i+1}BD^i \end{bmatrix}, \quad (3.6)$$

where

$$\begin{aligned} \Phi_k = & \sum_{i=0}^{r-1}(D^d)^{i+k+2}CA^iA^\pi + D^\pi C(A^d)^{k+2} \\ & - \sum_{i=0}^{k}(D^d)^{i+k}C(A^d)^{k-i+1}, \ k \geq 0. \end{aligned} \quad (3.7)$$

Proof. First, notice that from conditions $BD^{\pi}C = 0$, $BDD^d = 0$, we have that $BD^{\pi} = B$ and $BC = 0$. If we split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & BD^{\pi} \\ C & D^2D^d \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & DD^{\pi} \end{bmatrix},$$

we have $QP^2 = 0$ and $QPQ = 0$. Also, we have that matrix Q is s-nilpotent, and therefore $Q^d = 0$. Applying Corollary 2.2 we get

$$M^d = P^d \sum_{i=0}^{s-1} (P^d)^i Q^i + (P^d)^2 \sum_{i=0}^{s-1} (P^d)^i Q^i P - P^d. \tag{3.8}$$

Since $BD^{\pi}C = 0$ and $BD^{\pi}D^2D^d = 0$, matrix P satisfies the conditions of Corollary 3.1 and, after computing, we get

$$(P^d)^i = \begin{bmatrix} (A^d)^i & (A^d)^{i+1}BD^{\pi} \\ \Phi_{i-1} & (D^d)^i + \Phi_i BD^{\pi} \end{bmatrix}, \text{ for all } i \geq 1, \tag{3.9}$$

where Φ_i is defined by (3.7). After substituting (3.9) into (3.8) and computing the sum (3.8), we get (3.6). □

Theorem 3.4 *Let M be given by* (1.1). *If $BD = 0$, $D^{\pi}CA^2 = 0$, $D^{\pi}CAB = 0$ and $D^{\pi}CBC = 0$, then*

$$M^d = \begin{bmatrix} A^d + (A^d)^3BC + (A^d)^4BCA & (A^d)^2B + (A^d)^4BCB \\ \Psi_0 + \Psi_2BC + \Psi_3BCA & D^d + \Psi_1B + \Psi_3BCB \end{bmatrix}, \tag{3.10}$$

where

$$\Psi_k = \sum_{i=0}^{r-1} (D^d)^{i+k+2}CA^iA^{\pi} - \sum_{i=0}^{k} (D^d)^{i+1}C(A^d)^{k-i+1}, \ k \geq 0. \tag{3.11}$$

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & BD^{\pi} \\ DD^dC & D \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 0 \\ D^{\pi}C & 0 \end{bmatrix}. \tag{3.12}$$

Since $BD = 0$,$D^{\pi}CA^2 = 0$, $D^{\pi}CAB = 0$ and $D^{\pi}CBC = 0$, we have $QP^2 = 0$ and $QPQ = 0$. Moreover, $Q^2 = 0$. Applying Corollary 2.2, we get

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP. \tag{3.13}$$

Matrix P satisfies the conditions of Corollary 3.1, so we get

$$(P^d)^i = \begin{bmatrix} (A^d)^i & (A^d)^{i+1}B \\ \Psi_{i-1} & (D^d)^i + \Psi_i B \end{bmatrix},\ i = 1,2,3. \tag{3.14}$$

Substituting (3.14) into (3.13) we obtain (3.10). □

Remark 1) If the last condition $D^\pi CBC = 0$ from Theorem 3.4 is replaced with two weaker conditions $D^\pi CBCA = 0$ and $D^\pi CBCB = 0$, then matrices Q and P, defined by (3.12), satisfy the conditions of Corollary 2.3. Therefore, we have the following representation for M^d:

$$M^d = \begin{bmatrix} A^d + (A^d)^3BC + (A^d)^4BCA + (A^d)^5BCBC & (A^d)^2B + (A^d)^4BCB \\ \Psi_0 + \Psi_2BC + \Psi_3BCA + \Psi_4BCBC & D^d + \Psi_1B + \Psi_3BCB \end{bmatrix},$$

where, for all $k \geq 0$, Ψ_k is defined by (3.11).

2) If conditions $D^\pi CA^2 = 0$, $D^\pi CAB = 0$ and $D^\pi CBC = 0$ are replaced with stronger conditions $D^\pi CA = 0$ and $D^\pi CB = 0$, we have that $BCA = BD^\pi CA = 0$ and $BCB = BD^\pi CB = 0$. Hence, we get the representation from Theorem 2.7 [8] as a corollary of Theorem 3.4.

4 Numerical examples

In this section we give two examples as illustrations of Theorems 3.1 and 3.2. In the following example a 2×2 block matrix M is given, which does not satisfy the conditions from [6, 7, 8]. The representation for M^d is obtained applying Theorem 3.1.

Example 4.1 Let M be a matrix of the form (1.1), where

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 3 & 0 & 5 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 3 & 3 & 3 & 3 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $BC \neq 0$, representations for M^d from [6, 7, 8] fail to apply. After calculating, we get that $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$.

Hence, the conditions of Theorem 3.1 are satisfied and after applying it we have that

$$M^d = \begin{bmatrix} \frac{645}{10892} & \frac{1419}{5446} & \frac{327}{2723} & \frac{645}{10892} & \frac{99}{10892} & -\frac{20}{2723} & \frac{99}{10892} & -\frac{2851}{5446} \\ -\frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1433}{10892} & \frac{115}{5446} & \frac{645}{2723} & -\frac{1433}{10892} & -\frac{2525}{10892} & -\frac{40}{2723} & -\frac{2525}{10892} & -\frac{2979}{5446} \\ \frac{645}{10892} & \frac{1419}{5446} & \frac{327}{2723} & \frac{645}{10892} & \frac{99}{10892} & -\frac{20}{2723} & \frac{99}{10892} & \frac{2595}{5446} \\ -\frac{209}{10892} & -\frac{1549}{5446} & \frac{236}{2723} & -\frac{209}{10892} & \frac{2045}{10892} & \frac{302}{2723} & \frac{2045}{10892} & \frac{299}{5446} \\ \frac{633}{10892} & -\frac{1875}{5446} & -\frac{363}{2723} & \frac{633}{10892} & \frac{1389}{10892} & \frac{297}{2723} & \frac{1389}{10892} & \frac{267}{5446} \\ \frac{1475}{10892} & -\frac{2201}{5446} & -\frac{962}{2723} & \frac{1475}{10892} & \frac{733}{10892} & \frac{292}{2723} & \frac{733}{10892} & \frac{235}{5446} \\ -\frac{251}{10892} & \frac{537}{5446} & \frac{544}{2723} & -\frac{251}{10892} & -\frac{1609}{10892} & \frac{50}{2723} & -\frac{1609}{10892} & -\frac{2403}{5446} \end{bmatrix}.$$

The next example describes a 2×2 block matrix M, for which M^d can not be derived from the conditions given in [5]. However, we can apply Theorem 3.2 to obtain M^d.

Example 4.2 Let M be a matrix given by (1.1), where

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 5 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 3 & 0 & 5 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

We get that $CB \neq 0$, so we can not apply a representation for M^d from [5]. It can be checked that $BCA = 0$, $DCA = 0$, $CBC = 0$ and $CBD = 0$. Therefore we can apply Theorem 3.2 and we get

$$M^d = \begin{bmatrix} 0 & \frac{5}{68} & 0 & -\frac{25}{136} & -\frac{93}{544} & 0 & \frac{93}{272} & -\frac{93}{544} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{5}{68} & 0 & -\frac{25}{136} & \frac{43}{544} & 0 & -\frac{43}{272} & \frac{43}{544} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{5}{68} & 0 & -\frac{93}{136} & -\frac{25}{544} & 0 & \frac{25}{272} & -\frac{25}{544} \\ 0 & \frac{3}{34} & 0 & -\frac{15}{68} & -\frac{15}{272} & 0 & \frac{15}{136} & -\frac{15}{272} \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{8} \\ 0 & \frac{5}{68} & 0 & \frac{43}{136} & -\frac{25}{544} & 0 & \frac{25}{272} & -\frac{25}{544} \end{bmatrix}.$$

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